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ERROR STUDY OF ORBITAL PREDICTIONS

J. Boddy and M. Grossberg

Radio Corporation of America
Aerospace Communications and Controls Division
Burlington, Massachusetts

FINAL REPORT

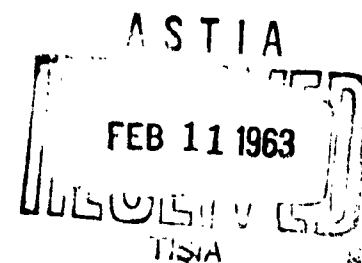
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20 April 1962

Prepared for

SPO 496L

AIR FORCE COMMAND AND CONTROL DEVELOPMENT
DIVISION
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS



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FOREWORD

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ABSTRACT

This study analyzes the statistics of sensor data obtained from the observations of artificial satellites. The effect of these statistics on the estimation of the orbital elements has been evaluated.

Several techniques for removal of the actual orbit from the observations to derive residuals have been investigated. The residuals have been obtained using both polynomial fitting with and without autoregressive scheme. From the satellites considered, no consistency was apparent in the magnitude of the variances of the individual parameters. However, a consistency was noted in the skewness of the residual. Computer programs have been developed for the determination of the statistics of the orbital elements from observation data. The study shows that the variances of the orbital elements are functions of the variance of the observations, the time of observation and description of the motion of the satellite. Furthermore the variance of any orbital elements is minimum if the sum of the variance of the observations at a given time is minimum for a specific computational procedure of the orbital element from observations.

The behavior in the statistics of the orbital elements exhibits a consistency in the measure of skewness and excess. Since these quantities are normalized with respect to the standard deviations it appears that a bias is introduced in the computation procedure. Thus, the problem of obtaining accurate position prediction of artificial satellites is one of interplay between the quality of observations and an analytical description of the motion of the satellite. How to trade these factors depends upon the time allowed for deriving the orbit from observations.

INTRODUCTION

This final report summarizes the results of a study on the error of orbital estimation from observed data performed by the Radio Corporation of America for SPO 496L Space Track R&D facility under Contract AF19(604) -8041.

Determining accurate orbits for an artificial satellite has presented a new problem in celestial mechanics. To imply that accurate positional information of an orbiting body can be obtained by more accurate observation is only partially correct. There is need for an analytical theory that would represent the motion of the vehicle with an accuracy consistent with the accuracy of the observations.

This study was concerned mainly with the accuracy of the observations and their effects on the orbital elements. More specifically, the study was devoted to

- (1) The analysis of the statistics of satellite tracking observations using data from tracking and surveillance radars and Baker-Nunn cameras,
- (2) The investigations of the statistical and dynamical assumptions governing the orbit computation based on the Herrick-Gibbs method and differential correction,
- (3) The analysis of the error propagation from observations to orbital elements following the Herrick-Gibbs method,
- (4) The investigation of intermixing different types of data and criteria.

Determination of the statistical quality of satellite tracking observations was conducted by gathering satellite tracking data essentially from the Millstone Hill Radar and AN/FPS-49. This analysis is contingent on the removal of the actual orbit from the observations. Several techniques were employed. Section 1 of this report discusses these methods; namely, simulation, differential correction, and polynomial smoothing with and without an autoregressive scheme. Each of these methods requires great care in its application, and the statistics gathered should not be regarded as absolute.

The orbit removal technique depends upon assumptions. To determine whether the assumptions adequately describe nature, it is necessary to look at the results of the orbit removal-statistical analysis procedure. There are two types of tests that may be applied. First, it may be noted whether the statistics are self-consistent, i. e., is a different statistical behavior for a given sensor's observations obtained on different satellites?

The second test is the "how well does it work?" type. Since the end product of the statistical analysis is to be a scheme for precision determination, estimates of sensor statistics may be used to construct an "optimum estimator," and to test the orbit so estimated against whatever high quality observations are available.

A large number of restrictions are placed on the use of differential correction. Without drag correction the observational span must be restricted to a few days, and, in any case, to a few weeks. A very limited number of observations from each sensor must be used and the relative quality of observations from different sensors must be known. Furthermore, a fairly large number of observations is necessary in order to obtain statistically stable residuals.

These restrictions, of course, greatly limit the number of satellites to which this technique can be applied. However, there will probably be some satellites with sufficient concentration of data. For these satellites, the amount of computation necessary is large, especially if the iterative procedure, suggested in subsection 1.2.1, is applied. An alternative approach to the orbit removal problem is polynomial smoothing which requires much less computation.

The simpler method of polynomial smoothing requires concentrated data. This method, which has been applied to Millstone data, has been reasonably successful. However, two procedures must be noted. In the first place, the observations to be reduced have to be taken over a time interval limited by the accuracy of the satellite theory. For example, if it is desired to analyze range observations from Millstone which have a standard deviation on the order of one kilometer, then these observations, plus any others which are used to establish the orbit, must cover a time span less than that over which the satellite theory is accurate to, say, one-half kilometer. This time span, even for a high satellite, is probably not much more than a few days at the most.

Secondly, using observations from different sensors without properly weighting them should be avoided. Since the statistical analysis to be performed is a prerequisite to the determination of weights, it seems best to perform the correction on observations made by a single sensor.

A disadvantage of the polynomial fitting is that, in the topocentric cartesian coordinates, all the parameters are smoothed by a same degree polynomial. The transformation to cartesian coordinates is an attempt to obtain uncorrelated residuals. With the autoregressive scheme employed in conjunction with the polynomial, the smoothing is done directly on the radar parameters. Each one can then be fitted by a different optimum degree of polynomial. The satellite observations handled by

this method indicate that the residuals of the radar parameters were uncorrelated.

In the process of this investigation it was found that the differential correction procedure used for astrodynamical data reduction is to be regarded as a method of solution of the unweighted least square problem (Ref. 2). It was also found that the statistics of the residuals exhibit non-zero third and fourth moments. It can be shown, using the fitting of observations by a trend and autoregressive scheme, that best estimators for predicting orbital elements can be obtained with independent estimators of the observations (Ref. 10). Expressions to obtain the statistics of the osculating elements and error propagation have been derived analytically. Their usefulness is limited due to the cumbersome formula. A computer program has been devised to derive the statistics of the osculating elements. This program utilizes the statistics of the observations, and the Herrick-Gibbs method. It gives the moments of the osculating elements.

SECTION 1

SMOOTHING AND DATA ANALYSIS

To determine the statistical character of electronic or optical observations of artificial earth satellites, it is necessary to isolate the observational errors from the "ideal" observations. The observations may be of slant range, azimuth and elevation, or of right ascension and declination. Therefore, one must determine, implicitly or explicitly, the true orbit of the satellite, calculate from this orbit the values which each sensor should have observed (the ideal observations), and subtract the ideal from the actual observations to obtain the errors.

The ideal observations may be determined by standard astrodynamical methods if the satellite's orbit is known. However, in order to use an orbit to obtain accurate observational errors, it is necessary to know the orbit with extremely great precision. Since the orbit is usually determined by applying a differential correction reduction on various observations, certain procedures must be avoided in this reduction or the "observational errors" obtained become meaningless.

1.1 DETERMINATION OF PARAMETRIC STATISTICS

1.1.1 DESCRIPTION OF METHODS

The inherent statistics associated with any observation parameter obtained from the sensing equipment should be fully understood, together with a knowledge of their magnitudes, limitations and model assumptions presumed.

These statistics could be obtained either from a data analysis study of a simulated target or from actual satellite observations. Both techniques, unfortunately, have their own advantages and disadvantages.

1.1.2 SIMULATION STUDY

Parameter statistics are sometimes gathered by means of simulated targets. For electronic sensors, signals representing those expected from an actual satellite may be introduced at the front end of the receiver. When this is done, the orbit removal problem becomes trivial, for the ideal observations are determined by the "orbit" which was simulated.

Besides reducing the problem of orbital removal to a purely numerical one, the technique of target simulation has several other important advantages. As much statistical data as desired may be gathered, merely by replaying the target through the sensor-computer system. Further, the quality of the observations may be calculated as a function of signal strength. Another important characteristic is that simulation offers a powerful tool for separating the random and systematic components of error.

The chief disadvantage of simulation is that one cannot be sure all sources of error are included. Propagation errors, ground reflections and target scintillation are usually ignored. Furthermore, special equipment used for generating the signal may introduce unwanted errors. In spite of these limitations, it is felt that statistical results gathered from a sophisticated simulation program should provide enough information on the character of observational errors to greatly simplify the analysis of tracking data. Unfortunately, extensive simulation has been carried out for very few sensors.

1.1.3 TRACKING DATA ANALYSIS

To obtain meaningful statistical information from actual satellite observations, a large number of observations must be available. However, because of the errors, no possible satellite orbit will exactly satisfy all of the observations. Thus, in order to estimate the elements of the orbit, some criterion is needed for closeness of fit of actual and ideal observations. This criterion is related to the mathematical model adopted for the errors, in the sense that different models give rise to different criteria for efficient estimators of the elements, and the different estimators will usually give rise to different sets of observational residuals. If an error model which does not adequately describe the character of the errors is used, the residuals obtained will be significantly at variance with both the adopted and the true model. The analysis of satellite tracking data may be regarded as an iterative procedure. In this procedure an orbit is determined which minimizes a criterion based on the assumed error model and the residuals are tested against the model. A good first approximation is needed, since "convergence" is rarely assured.

1.2 DIFFERENTIAL CORRECTION

1.2.1 DESCRIPTION OF MODEL FOR DIFFERENTIAL CORRECTION

The correct selection of an error model to best describe the character of the errors is extremely important if the residuals are to have significant meaning. The simplest error model consists of the following assumptions:

- (1) All observational errors are Gaussian random variables
- (2) All errors are uncorrelated
- (3) All errors have zero means (observations are unbiased).

With this model, the most "efficient" estimator is the rigorous least squares estimator. An exacting discussion on the most "efficient" estimator is given in Ref. 2. The estimator picks out that orbit which minimizes the sum of the squares of the weighted residuals, where the weight for a given residual is the inverse of the standard deviation of the observational error. To actually compute the orbital element estimates, a nonlinear system of equations must be solved. Differential correction is an iterative technique for solving these equations. (See Ref. 1, 2 and 3.)

As mentioned, the weights to be used with a differential correction depend upon the standard deviations of the errors. Since these standard deviations are to be obtained by the statistical analysis which follows the orbit removal, the need of an additional assumption describing the relative sizes of the error is apparent. This assumption is crucial, and will be discussed in detail in subsection 1.2.3.

1.2.2 THEORETICAL LIMITATIONS OF DIFFERENTIAL CORRECTION

To obtain meaningful residuals, the differential correction procedure must not introduce theoretical errors, i. e., the correction must select the true least squares estimate of the orbit to rather high accuracy. In particular, this requirement imposes the following restrictions on the procedure:

- (1) Quantities derived from observations must be corrected if their uncertainty gives rise to significant variance in the residuals.
- (2) The time span of the observations must be no greater than the span over which the satellite theory is of "sufficient accuracy"
- (3) The loss of accuracy involved in computing partial derivatives from a Keplerian ellipse assumption must be investigated

- (4) Great care should be taken in the numerical processes, especially the matrix inversion.

The first restriction can best be explained by a numerical example. One effect of atmospheric drag is to reduce the period of a satellite. A typical value of this reduction is 10^{-8} days per revolution per revolution. Suppose a preliminary estimate of the drag parameter has been made which is sufficiently uncertain as to make the value of the period reduction uncertain up to 10^{-9} days/rev². This, in turn, causes a variable time uncertainty whose magnitude is given approximately by 10^{-9} times the square of the number of revolutions from the center time. For two weeks of observations this amounts to a maximum time error of about 10^{-5} days, hence a maximum position error of about 6 or 7 kilometers. Such an error is larger than the standard deviation of most observations; therefore, it is necessary to correct the drag parameter as well as the orbital elements. Indeed, with an uncertainty of 10^{-9} , drag corrections are necessary whenever the observations span more than about three days, or even less time if high-accuracy optical observations are to be analyzed.

The second restriction refers to the state of the art. If exact initial conditions of the satellite at some epoch are given and the position of the satellite can be computed to an accuracy of 0.2 km for only one week in the future, then the observational span should be restricted to about two weeks or less. This restriction, unlike the other three, cannot be removed by improving the differential correction program.

The third restriction is, perhaps, less important than the others. The differential correction procedure employs the partial derivatives of ideal observations with respect to the parameters which are to be corrected. These derivatives may be computed either numerically or analytically. When analytical differentiation is employed, the

derivatives are computed on the basis of a two-body Keplerian orbit. This introduces a second order error into the estimates. The fourth restriction is self-explanatory.

1.2.3 STATISTICAL LIMITATIONS OF DIFFERENTIAL CORRECTION

The properties which determine the degree of sophistication necessary in the differential correction program have been examined. The type of data to be used with the program is analyzed in the light of the assumptions on the error model.

The assumption of Gaussian error distribution is not too critical. Actual distributions of errors will not differ too much from the Gaussian, and for these distributions the least squares estimator will still be highly efficient. The assumptions of unbiased and statistically uncorrelated observations are more important. If these assumptions are not satisfied, and in practice they may not be, the accuracy of the estimated orbit is greatly reduced. For this reason, whenever there is a suspicion of such bias or correlation, the amount of data used from each pass over each sensor should be restricted to a few points. In this way, the bias may be treated as an addition to the random error, and the effects of serial correlation are greatly reduced.

The assumption on the relative quality of observations is critical. Briefly, the accuracy of an improperly weighted least squares estimation is determined by the more grossly overrated observations. Thus, if the data include several observations of very poor quality, these observations must have corresponding low weights attached or they seriously distort the residuals.

For this reason, if good a priori information on the relative quality of the observations is not available, the differential correction should be regarded as an iterative procedure, whereby the residuals are examined

to determine new weights for the next correction. Such a procedure has no guarantee of convergence.

1.3 POLYNOMIAL SMOOTHING

The technique of polynomial smoothing can be used when compact sets of satellite observations are available. The polynomial smoothing consists of finding those time polynomials of a specified degree which best fit the azimuth, elevation and range parameters in the least square sense. The polynomial smoothing technique is based on the same assumptions that apply to differential correction. The former technique gives a less optimum estimate but requires much less computation than the latter. Polynomial smoothing is applicable only to radar tracking data, whereas the differential correction technique may be applied to a wide variety of observations. The observations need not be taken at equal intervals of time.

1.3.1 DESCRIPTION OF POLYNOMIAL SMOOTHING

The polynomial smoothing technique consists of finding that polynomial of a given degree which best fits radar observations of a given type in the least squares sense. For example, to fit radar range with a second degree polynomial, three quantities – C_0 , C_1 , C_2 – must be found which minimize the expression, S:

$$S(C_0, C_1, C_2) = \sum_t (R_o(t) - C_0 - C_1 t - C_2 t^2)^2 W_{R(t)}^2$$

where $R_o(t)$ is the observed range at time, t ; $W_{R(t)}$ is the inverse of the standard deviation; and the summation extends over all the observation times. This problem has a unique solution unless, of course, there are fewer observations than constants to be determined. The solution reduces to a linear system of equations. The application of this technique to the problem of orbit removal is presently discussed.

The range observations are the sums of the ideal observations plus the observational errors. If the errors, in turn, are sums of a constant bias, b , plus a purely random error, the observations may be expressed as follows:

$$R_o(t) = R_I(t) + b + R(t)$$

where $R(t)$ is the random error. Subsection 1.3.2 shows that, with a proper choice of the polynomial degree, the curve fitted to the observations will closely approximate the sum of the ideal observations and the bias. It is further shown in subsection 1.3.3 that this curve will not approximate the random errors very well. Thus, the technique of polynomial smoothing separates the observations as follows: the curve fitted to the observations approximates the sum of the ideal observation and the bias; the residuals or differences between the observations and the curve approximate the random errors.

1.3.2 THEORETICAL LIMITATION

The equations for the polynomial coefficients are linear, thus the estimated coefficients are linear functions of the observations. This implies that the curve of best fit to the observations is given by the sum of the curves of best fit to the ideal observations, bias, and random errors. Therefore, each component of the observations may be analyzed separately.

The curve of best fit to a constant bias is simply that bias itself. For the ideal observations, assume that a polynomial of degree ' p ' is to be fitted to range observations covering a time span, $2T$. Let t_o be the midpoint of this span. The ideal range in a truncated Taylor Series expansion about t_o is:

$$R_I(t) = \sum_{r=0}^p \frac{R_I^{(r)}(t_0) (t-t_0)^r}{r!} + \frac{R_I^{(p+1)}(t') (t-t_0)^{p+1}}{(p+1)!}$$

The first term on the right-hand side can be fitted exactly by a polynomial curve of degree p . The second term, the remainder, involves the $(p+1)^{st}$ derivative of the ideal observation evaluated at some time, t' , between t_0 and t . To determine how closely this term can be fitted is an extremely difficult task, however its magnitude can be analyzed. The magnitude of the remainder is less than:

$$\frac{|R_I^{(p+1)}|_{\max} T^{p+1}}{(p+1)!}$$

The derivate is maximized over the time interval, $t-t_0 \leq T$. Thus, if the radius of convergence of the Taylor Series is greater than T , (Taylor Series converge uniformly), one can find a p such that the remainder terms are sufficiently small. In practice, p can be chosen so that the magnitude of the remainder term is a small fraction (one-tenth or less) of the expected standard deviation of the observational errors. Examples of the magnitudes of these standard deviations have been computed and are given in Appendixes A and B.

To calculate the magnitudes of the derivatives, at least two schemes are possible. They could be computed from the equations of ballistic motion. In the LaGrangian equations of motion, the second derivatives of range, azimuth and elevation are expressed in terms of their zeros and first derivatives. An estimate of the zeros and first derivatives can be obtained from the record. Differentiation of the LaGrangian equations the proper number of times yields the desired result. This approach rapidly gets out of hand, however, and its use is not recommended.

The second scheme is more simple to perform but is subject to numerical computation errors from an assumed orbit. An approximate orbit from the record is determined. From this orbit the radar parameters at equally spaced time intervals are calculated. The derivatives of these radar parameters are inferred. Since the orbit used is only approximate, and the numerical differentiation introduces errors, the resulting derivatives should be used with a safety factor on the maximum allowable remainder.

1.3.3 STATISTICAL LIMITATIONS

The polynomial coefficients are computed on the basis of noisy observations therefore they are subjected to random fluctuations. For the same reason, the fitted curve is also subjected to fluctuations. If the fluctuations in the curve are of the same order of magnitude as the fluctuations in the observations, the residuals will bear little resemblance to the observational errors. When a large number of observations is employed, the fluctuation in the curve becomes quite small. In Appendixes A and B formulas and graphs for the ratio of the standard deviation of the curve to the deviation of the observations are presented as a function of the degree of the polynomial, number of points, and time. These graphs are derived on the basis of the assumptions that the observations are uncorrelated, stationary, and equally spaced in time. The non-stationary state of the cartesian coordinates does not make them strictly applicable, but they are nevertheless approximately correct.

A numerical example explains the reason the radar coordinate of lowest standard deviation gives rise to distorted residuals. The following table shows the true values of the satellite in radar and cartesian coordinates and the radar standard deviations at a particular instant of time:

R	E	A	σ_R	σ_E	σ_A	X	Y	Z
800	30°	45°	2.5	0.2573°	0.1719°	490	490	400

Now the standard deviations in X, Y, Z may be found as follows:

$$\begin{aligned}\sigma_X^2 &= (\sigma_R \cos E \cos A)^2 + (R \sigma_E \sin E \cos A)^2 + (R \sigma_A \cos E \sin A)^2 \\ &= 4.744 = (2.178)^2\end{aligned}$$

Similarly,

$$\sigma_Y^2 = (2.178)^2 \text{ and } \sigma_Z^2 = (1.429)^2$$

Assume next that the standard deviations of the curves fitted to the cartesian coordinates are each equal to one-twentieth of the standard deviations of the observations. The table below shows these deviations and the corresponding deviations in the curves fitted to the radar coordinates:

$\frac{\sigma_{Xc}}{\sigma_X}$	$\frac{\sigma_{Yc}}{\sigma_Y}$	$\frac{\sigma_{Zc}}{\sigma_Z}$	$\frac{\sigma_{Rc}}{\sigma_R}$	$\frac{\sigma_{Ec}}{\sigma_E}$	$\frac{\sigma_{Ac}}{\sigma_A}$
0.109	0.109	0.071	0.101	0.00588°	0.00901°

Then:

$$\sigma_{Rc}/\sigma_R = 0.04 \quad \sigma_{Ec}/\sigma_E = 0.103 \quad \sigma_{Ac}/\sigma_A = 0.052$$

Thus, although the curves were fitted equally well to the three cartesian coordinates, their resulting deviations in the radar parameter have become unequal and distorted.

1.3.4 DEGREE OF POLYNOMIAL

A. A Priori

A priori knowledge of the degree of polynomial required to fit a given set of observations can be obtained from the determination of polynomial fitting to the true orbital path for the given length of observation time. Although the true trajectory is unknown, a family of trajectories about the ideal value can be investigated to determine the optimum degree of polynomial required. The magnitude of the errors obtained by approximating the ideal trajectory with any polynomial is indicated in Appendixes A and B.

For most satellites the Millstone Hill radar parameters can be approximated by a fourth order polynomial for observation times up to 5-1/2 minutes. Table A-1 in Appendix A indicates the time spread of the amount of data acceptable for various deviations in the fit of different degree polynomials to the ideal path of satellite motion.

B. A Posteriori (Orthogonal Polynomials)

Determining the polynomial degree may also be done a posteriori with the use of orthogonal polynomials. The polynomials are described in Ref. 4-~~and~~ only certain properties will be mentioned here.

The most important property of these polynomials is that they may be used to determine the polynomials of best fit of several different degrees, with a minimum of duplication. Closely connected with this property is a simple statistical test (F-test) which may be made on the difference between the variance of the residuals after fitting polynomials of successive degrees to the data. In this way, an a posteriori test of the significance of the p^{th} order polynomial term may be made.

Use of orthogonal polynomials has certain computational advantages over the usual polynomials. The advantages are most pronounced when several different polynomial degrees are to be tested. These advantages are nullified, if an a priori determination of the degree has been made or if the observations are not equally spaced in time, or if the observations are to be weighted unequally.

C. Variate Difference

The technique of applying the variate difference methods to the observations is considered in detail in Ref. 1. Basically it determines the degree of the polynomial fit and also estimates the variance of the random component by considering the stability of the forward differences of the parameters. It assumes that the radar reports are composed of a systematic and a random component:

$$X_t = \sum_{k=0}^{k=p} A_K^t K + \epsilon_t$$

where p is the order of the time polynomial to be estimated. Estimates of the variance of the forward differences are obtained. This variance of the random component decreases up to the pth difference and then the variance should stabilize except for random fluctuations. These random fluctuations could be due to a harmonic periodicity present in the residuals.

1.4 AUTOREGRESSIVE SCHEME

The residuals obtained after fitting the optimum degree of polynomial still exhibit a sinusoidal tendency. It can be shown that the fitting of the optimum order polynomial plus an autoregressive scheme will better describe the error process. In addition, this method gives rise to independence between the observations for the cases considered (Ref. 10).

1.4.1 DESCRIPTION OF AUTOREGRESSIVE TECHNIQUE

With the determination of the order p of the polynomial and its least square coefficients, the residuals X_t can be fitted by an autoregressive scheme. The autoregressive scheme considered in Ref.10 is of the second order:

$$\Delta X_t = \sum_{k=1}^{k=2} b_{t-k} \Delta X_{t-k}$$

The coefficients b_{t-1} , b_{t-2} , are consistent estimators in the least square sense. In addition to being consistent estimators, these least square estimators are asymptotically best estimators. The solution to the autoregressive difference equations has a harmonic solution which could alter the behavior of residuals obtained from polynomial fitting only.

The solution is given by:

$$X_t = p^t (C_1 \cos \theta t + C_2 \sin \theta t)$$

$$X_t = C_1 \beta_1^t + C_2 \beta_2^t$$

depending on whether the roots of $\beta^2 - b_{t-1}\beta - b_{t-2} = 0$ are imaginary or real.

The particular solution takes the form:

$$\sum_{j=0}^{j=\infty} A_j \epsilon_{t-j+1}$$

so that initially errors are independent. Once introduced into the process, they forever exert their influence. For this reason, it was expected that residuals would be uncorrelated, the dependence tied up in the initial observations and remaining uncorrelated throughout the process, and therefore tending to diagonalize the covariance matrix. An estimate of the variance of the autoregressive scheme is

$$\sigma_{X_t}^2 = \sigma_{\epsilon_t}^2 \frac{(1-b_2)}{(1+b_2)(1-b_2^2 - b_1^2)}$$

The residuals after trend and autoregressive fitting were considered elements of ϵ_t and assumed to be a stationary process up to second order. Whereas only the second degree autoregressive scheme is indicated in Ref. 10, the degree actually required can be estimated utilizing similar methods to those used to determine the degree of the polynomial via a variate difference method.

1.5 EFFECT OF REFERENCE SYSTEM

There are various systems of coordinates in which the observation can perhaps be smoothed better than in another system. The systems of radar parameters do not behave like low order polynomials for any length of time. In order to carry out polynomial smoothing on these coordinates, it is necessary to resort to polynomials of very high degree - perhaps, ten or twelve or more for a typical 5-minute pass. It is not recommended that such high order polynomials be used, as they require the solution of large scale systems of simultaneous equations. Such solutions consume much time and storage and are of limited accuracy. Another objection to high order polynomials is the loss of statistical stability involved. Furthermore both a priori and a posteriori techniques of estimating polynomial degree become questionable.

Two solutions to this problem are known. Either a sequence of functions may be found (linear and nonlinear) which fit the observations better than polynomials (autoregressive scheme perhaps) or the observations may be transformed to a form in which they are easily fitted by polynomials. Unless differential correction is carried out it seems quite difficult to measure the truncation errors associated with any smoothing other than polynomial fitting. Furthermore, most smoothing functions will give rise to nonlinear systems of equations which will involve a great deal of machine computation.

1.5.1 POLAR VERSUS TOPOCENTRIC CARTESIAN REFERENCE SYSTEM

Although the radar coordinates range, elevation and azimuth do not behave like low order polynomials, the cartesian inertial and the topocentric cartesian coordinates of the satellite do. Thus the transformation from radar to topocentric coordinates produces records which are very amenable to polynomial smoothing. Appendix A shows that, for a wide variety of orbits, the truncation associated with a p^{th} degree polynomial is a function only of the time interval, and that for typical radar data, a fourth degree polynomial is sufficient.

The weights to be used with the cartesian coordinates depend upon the standard deviations of the radar coordinates. Since these are unknown, an iterative technique is employed. This technique has been programmed for the IBM 7090 in Fortran language and performs the following computations.

Initially the standard deviation in range, azimuth, and elevation, and the polynomial degree are assumed. The observations are entered and transformed to cartesian coordinates. Weights for the cartesian coordinates are found on the basis of the assumed radar deviations and the formulas of linear error propagation. Then normal equations for the polynomial coefficients are solved, residuals are computed and

transformed into residuals in range, azimuth, and elevation. The standard deviations of these residuals are computed and compared with the assumed values. If the discrepancy is large, the procedure is repeated starting with the computation of weights, replacing the radar deviations with the values just computed.

This program has been applied to several records. In all cases, convergence has been achieved after two or three iterations. The final residuals have not deviated significantly from the Gaussian (subsection 1.6.1). Serial correlations have been small for those coordinates with relatively large standard deviations, but have been significant when the deviations are small - usually in elevation. This need not contradict the hypothesis of statistical independence, due to the poor statistical stability in these residuals. In general, it is believed that the residuals in those radar coordinates with large standard deviations are close approximations to the actual random errors, while, for the other coordinates, the standard deviations are probably reasonable, but the residuals themselves are in error. An example of the statistical limitations and distortion of errors is indicated in subsection 1.3.3.

1.6 DATA ANALYSIS

1.6.1 RADAR DATA FROM MILLSTONE AND MOORESTOWN

Samples of satellite observations from two different sites, Millstone and the FPS-49 radar at Moorestown, were analyzed to determine the distribution of the errors, and their correlation. The data supplied by Moorestown were 10-second radar parameters for various satellites. These satellites included 1959 Epsilon 1 and Iota 1, 1960 Zeta 1, Zeta 2, and Nu 2, and 1961 Epsilon 1, Lambda 1, and Omega 1. The Millstone data were 6-second semi-smoothed radar parameters from 1960 Epsilon 1 and Iota 1. Various degrees of polynomials were fitted to the transformed cartesian coordinates. The statistics of the resulting

residuals of the radar parameters were obtained. From the satellites considered, there appeared to be no consistency in the magnitude of the variances of the individual parameters. A typical example is given for 1960 Epsilon 1 and shows the effect on the first four moments of fitting different degree polynomials for each of the radar parameters. In general, for range, azimuth and elevation the β values were between 2 and 3, this being a measure of the normality of the distribution. The values of γ obtained were consistently less than 1, i. e., the residuals exhibited a tendency toward a slight skewness to the right for the bias.

For the four orders of polynomials fitted to 1960 Epsilon 1 the covariance matrix remained relatively unchanged. It can also be seen that the covariance was not a diagonalized matrix; there was a strong correlation between range and the other two parameters. A usual assumption made about radar data is that the parameters are uncorrelated to facilitate easy inversion of a simple diagonal matrix. Whereas these residuals with a polynomial fitting exhibited a marked correlation, when the radar parameters are fitted with a polynomial plus an autoregressive scheme, as indicated in Ref. 10, the covariance matrix obtained can be assumed to be diagonalized. Also, the standard deviations of the radar parameters using the autoregressive scheme have been decreased. The residuals, after fitting the autoregressive scheme, are shown in Figures 9 to 11 of Ref. 10. There is an indication that a periodic trend in the residuals is still present.

The serial correlation coefficients of the radar parameters indicated that the residuals obtained were independent of the time of observation.

1960 EPSILON 1 MILLSTONE 9/6/60, n = 33

Degree of Polynomial	2nd	3rd	4th	5th
Range				
μ	0.022969696	-0.007393939	0.005303030	0.000848484
δ^2	4.9927945	5.127000000	4.843638531	4.767295187
$\Sigma(x - \mu)^3$	50.57373	19.8524800	67.4695200	38.3293800
$\Sigma(x - \mu)^4$	1753.81997	1755.10877	1527.42081	1455.01365
γ	0.150803582	0.056888168	0.210548501	0.122497143
β	2.22849048	2.10485766	2.04748192	2.01010781
δ	2.234456190	2.264287967	2.200826783	2.183413654
Elevation				
μ	0.004363636	0.001666666	-0.000818181	0.000424242
δ^2	0.702516312	0.647099843	0.318515093	0.268047437
$\Sigma(x - \mu)^3$	-9.7951800	-14.9717800	-0.2226700	-0.3515300
$\Sigma(x - \mu)^4$	43.40384	52.6969800	13.44975	9.68441
γ	-0.553387199	-0.956794399	-0.041204754	-0.084264689
β	2.83477103	4.14144446	4.37259861	4.44876366
δ	0.838162461	0.804425164	0.564371414	0.517232978
Azimuth				
μ	-0.004181818	0.010878787	0.008272727	-0.001363636
δ^2	1.273498937	0.707355593	0.705998218	0.701886406
$\Sigma(x - \mu)^3$	-10.23896	-1.1498300	-0.56014000	0.0924800
$\Sigma(x - \mu)^4$	109.17207	31.3011400	30.62124	28.76527
γ	-0.237006096	-0.064294637	-0.031410683	0.05231016
β	2.12362687	1.95961482	1.92077015	1.81594638
δ	1.128494101	0.841044346	0.840237001	0.837786611

VARIATION OF COVARIANCE MATRIX WITH DEGREE OF POLYNOMIAL FITTED

2nd Polynomial Fit

3rd Polynomial Fit

$$\begin{pmatrix} R & 4.9927945 & -0.5001442 & 0.4015138 \\ E & -0.5001442 & 0.702516312 & 0.0000188 \\ A & 0.4015138 & 0.0000188 & 1.273498937 \end{pmatrix} \begin{pmatrix} 5.127 \\ -0.2978848 \\ 0.4439724 \end{pmatrix} \begin{pmatrix} -0.2978848 & 0.4439724 \\ 0.647099843 & -0.0000186 \\ -0.0000186 & 0.707355593 \end{pmatrix}$$

4th Polynomial Fit

5th Polynomial Fit

$$\begin{pmatrix} R & 4.843638531 & -0.6147249 & 0.5402671 \\ E & -0.6147249 & 0.318515093 & 0.0000069 \\ A & 0.5402671 & 0.0000069 & 0.705998218 \end{pmatrix} \begin{pmatrix} 4.767295187 \\ -0.5945959 \\ 0.5376413 \end{pmatrix} \begin{pmatrix} -0.5945959 & 0.5376413 \\ 0.268047437 & 0.0000005 \\ 0.0000005 & 0.701886406 \end{pmatrix}$$

1.6.2 BAKER-NUNN SATELLITE TRACKING CAMERA

The Baker-Nunn tracking camera is especially designed for tracking artificial earth satellites (Ref. 6). The camera is briefly described as a three-axis Super-Schmidt $f/1$ camera, with a focal length of 50 cm, and a field of view 5 by 30 degrees, which is photographed on a 55-mm cinemascope film about one foot long. Faint satellites are tracked to obtain measurable images. When the tracking rate of the camera is the same as the rate of the satellite, the satellite appears as a point image on the film against a star trail background. The position of the satellite is then determined by measuring the centers of the central breaks of reference stars.

The number of observations in angular position that the Baker-Nunn camera can acquire on one pass of a satellite depends upon the satellite's altitude and brightness; but generally, two or three observations are taken. For satellites that are high and bright, such as Iota 1, it is possible to obtain ten sample observations in angular position. This is not a sufficient number of observations to make a reliable statistical analysis.

There are several possible schemes for obtaining more sample observations, one of which is to analyze the residues of the observations made on many passes of a given satellite with a particular Baker-Nunn camera; but each pass constitutes a different experiment. It is reasonable that, with each experiment, (in effect, measurements under different geometries) stationary statistics cannot be assumed valid.

A second scheme that might be suggested is an ensemble of stations making observations on a given satellite. This is not practically feasible. In addition, in analyzing the errors, the ensemble averages and the time averages would have to be assumed the same. In either scheme, the process is neither stationary nor ergodic.

Because one is dealing with an experiment which involves so few sample points and is devising experiments to increase the sample size, the assumption of stationarity or ergodicity is established. However, this is not valid. The approach taken by the Smithsonian Institute for establishing accuracies is quite therefore reasonable (Ref. 5).

The accuracies established for the Baker-Nunn camera were determined as follows: Observations from stations were compared with respect to a very accurately known orbit and the residues were analyzed.

Two types of data were analyzed: (1) field reduced observations and (2) precisely reduced observations. The field reduced data are measurements made by the observers at the stations and are subjected to personal error. The field reduced positions are accurate to within $2\frac{1}{2}$ minutes of arc in standard deviation and one-tenth second of time in standard deviation.

Precisely reduced data are field data sent via airmail to the Photo-reduction Division of the Smithsonian Institute in Cambridge, Massachusetts and analyzed under elaborate laboratory conditions. Precisely reduced data have an accuracy of two seconds of arc and 0.001 second of time in standard deviation.

1.6.3 MULTI-SITE INITIAL WEIGHTING

If an initial knowledge of the variance of the data can be estimated, then it can be utilized for quicker convergence in the Fortran program for the polynomial fitting technique as described in subsections 1.3.4 and 1.5.1. The following technique has the advantage of obtaining an estimate of the parameter variance of one site if the parameter statistics are known for another site. The assumption made is that the variances are unchanged when subjected to an orthogonal transformation. To carry out this weighting, it is first necessary to obtain true or expected

parameter values for azimuth, elevation and range. However, at any time, for any satellite, these values are not known. In Ref. 7, equations are given for the determination of rectangular coordinates for a satellite in a geocentric coordinate system. By reworking these expressions, the following procedure is obtained for prediction of the satellite parameters which should be observed at one radar at a given time, t , when the parameters are known for some other radar at that time, t . The known inputs are the positions of the two radar sites R_i, ϕ_i, θ_i , the geocentric coordinates of each radar site, and the radar parameters from both sites at corresponding times, $(S_i, h_i, A_i) = (\text{slant range, elevation, azimuth})$. (See Figure 1-1.)

The subscript merely denotes from which radar site these parameters have been observed. The basic equations for transferring the position coordinates of a satellite, as observed from one site into those as observed at a second site, are given as follows:

$$\beta_1 = \sin^{-1} \left[\frac{(R_1 \cos h_1)}{(R_1^2 + S(S + 2R_1 \sin h_1))^{1/2}} \right]$$

$$\alpha_1 = \frac{\pi}{2} - (h_1 + \beta_1)$$

The geocentric polar coordinates of the satellite are given by

$$r_s = [R_1^2 + S(S + 2R_1 \sin h_1)]^{1/2}$$

$$\cos \phi_s = \sqrt{1 - (\cos \alpha_1 \sin \phi_1 + \sin \alpha_1 \cos \phi_1 \cos A_1)^2}$$

and also

$$\theta_s = \theta_1 - \sin^{-1} \left[\frac{\sin \alpha_1 \sin A_1}{\cos \phi_s} \right]$$

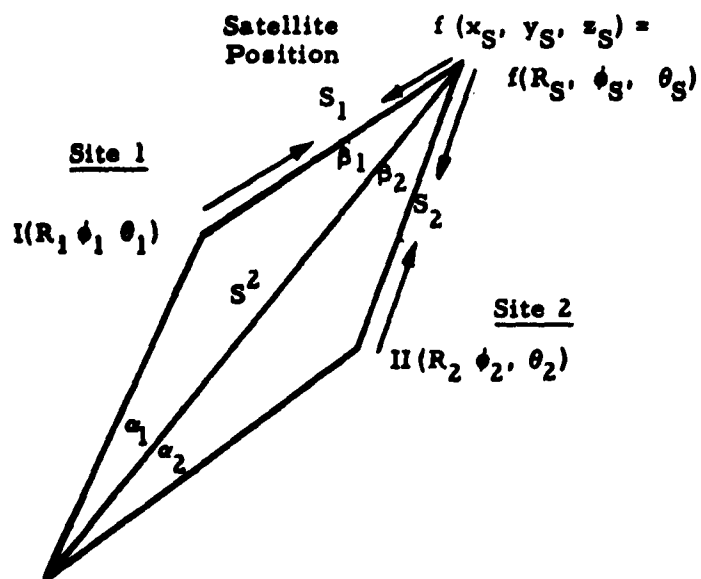


Figure 1-1. Simplified geometric representation for parameter prediction

After completing the transformation the geocentric cartesian coordinates are given by

$$\left. \begin{aligned} X_s &= r_s \cos \phi_s \cos \theta_s \\ Y_s &= r_s \cos \phi_s \sin \theta_s \\ Z_s &= r_s \sin \phi_s \end{aligned} \right\}$$

These geocentric cartesian coordinates can now be converted, as obtained from Site 1 data, into radar parameters as would be observed at Site 2.

$$S_2 = \left\{ R_2^2 - r_s^2 - 2 \left[x_s R_2 \cos \phi_2 \cos \theta_2 + y_s R_2 \cos \phi_2 \sin \theta_2 + z_s R_2 \sin \phi_2 \right] \right\}^{1/2}$$

$$h_2 = \sin^{-1} \left[\frac{r_s^2 - R_2^2 - S_2^2}{2 R_2 S_2} \right]$$

$$\beta_2 = \sin^{-1} \left(\frac{R_2 \cos h_2}{r_s} \right)$$

$$\alpha_2 = \frac{\pi}{2} - (h_2 + \beta_2)$$

$$A_2 = \cos^{-1} \left[\frac{\sin \phi_s - \cos \alpha_2 \sin \phi_2}{\sin \alpha_s \cos \phi_2} \right]$$

Thus, by considering the different radar sites and the preceding expressions, values from one radar site can be used to predict corresponding values for another site, say Millstone Radar. Also, observed values at Millstone will be used to predict values for another radar. For a given time interval then, four sets of compatible data can be obtained.

For a given time, t , an observed variance S^2 , can be obtained for these data. To accomplish this the following information is considered.

It is known that the errors between the observed values and the expected values are approximately Gaussian distributed, $N(0, \sigma_{R-\mu})$, where R is observed value and μ is the expected value.

$$\sigma^2_{R-\mu} = E([R-\mu] - E [R - \mu])^2$$

Thus,

$$\sigma^2_{R-\mu} = \sigma^2_R$$

Thus,

$$(R-\mu) \text{ is } N(0, \sigma_R)$$

For this reason, with S^2 known, bounds on σ^2 can be obtained by employing the X^2 distribution with $n-1$ degrees of freedom given by

$$X^2 = \frac{n S^2}{\sigma^2}$$

where n is the sample size.

These bounds on σ^2 can be used to obtain boundary equations (with an attached degree of confidence) for polynomial smoothing as discussed in subsection 1.3.1.

SECTION 2

ORBITAL COMPUTATIONS AND ERROR BEHAVIOR

Error analysis deals with the type of errors in the prediction and future ephemeris implied by the expected errors in the sensor data. This concerns the mathematical probability that the vehicle will be in an expected spacial position at a given time.

Error behavior can be studied as a linear or a nonlinear problem. The linear analysis is based on assumptions that the errors in the prediction are linearly related to the errors in the input data. This is realistic if the standard deviations of the data errors are extremely small and produce prediction functions with multivariate Gaussian distribution, with their covariance matrices related by simple matrix multiplication involving partial derivatives of prediction functions with respect to observations. The way the error analysis is performed depends upon the distribution of statistics assumed and the dynamical assumptions applied to the arc of trajectory during the period of observations. The first section of this report dealt with various techniques for smoothing these data. This section discusses the error behavior of the osculating elements and finally the ephemeris.

Statistical assumptions made about the radar parameters are that they are independent, Gaussian, and can be expanded in a Taylor Series and truncated after the linear terms without any loss in accuracy. Errors are also assumed to be non-serially correlated and time invariant.

While several of these assumptions might hold for some radar observations so that a linear error analysis could be used for the osculating

elements, the final elements would certainly not be independent but would exhibit marked correlation. Some results obtained in the previous section tend to indicate that most, if not all, of these assumptions should be considered open to suspicion. The only assumption that seems to be allowed is that the time of observation is known nearly exact.

2.1 OSCULATING ELEMENTS DETERMINATION

The initial point of the study of error behavior is the conversion of three smoothed vectors into one smoothed midpoint velocity vector by a method attributed to Herrick-Gibbs. This technique is employed by Space Track for the conversion of radar data (Ref. 7).

An alternative method is to obtain the velocity vector directly from the radar data. One of the useful by-products of the topocentric polynomial smoothing is a measure of the midpoint three components of velocity as coefficients of the first order terms in the polynomial and the acceleration as the second order terms. The statics of this velocity are also easily available. Now that a position and a velocity vector are both known for a common time, the osculating elements can be uniquely determined (see Appendix D).

The smoothed radar data exhibited a certain amount of skewness and non-normality and the radar parameters were definitely not independent. There appears to be no serial correlation of the parameters and it was not ascertained whether the covariances were time invariant. With this knowledge of the statistics of radar smoothed data it was decided to investigate the error behavior of the osculating elements (1) on the basis of linear independent errors and (2) on error behavior in which the first four moments of the input data were known with their cocumulants. The radar parameters were expanded in a Taylor Series about their means and the series truncated at fourth order terms. The method of propagation of errors was similar to that of Ref. 11. It quickly became

obvious that this was an unrewarding tedious task. The procedure was reduced to only the first two moments and the covariance. The necessary partial derivatives and formulas for the error prediction of the osculating elements are indicated in Appendix E.

2.2 PREDICTION DETERMINATION

Error behavior of the final prediction was analyzed in Appendix E by a method similar to that employed to compute the osculating element errors. The results obtained from the analysis of several satellites indicated that the osculating elements were strongly correlated and were not Gaussianly distributed (see subsection 2.3). Error propagation for the first four moments was carried out to evaluate the four moments of the cartesian coordinates but the covariance was approximated to second moments only. Only the first partial differentiations were carried out in Appendix E due to their lengthy nature. All of the partial derivatives have a periodic tendency which is non-decreasing with time. The main fault of lengthy prediction time is due to the partial derivatives of the spatial position with respect to the semi-major axis $\partial \underline{r} / \partial a$.

Although this derivative was also periodic, the magnitude of this period increased with time. When higher order derivatives are considered they will be increasing with higher orders of time. Thus, prediction error will increase with time. It might be predominant compared with the errors introduced when considering a simple Keplerian system, ignoring the effects caused by earth's oblateness, drag and several other second order effects on the satellite.

2.3 NUMERICAL ERROR BEHAVIOR OF OSCULATING ELEMENTS

Samples of Millstone 6-second radar data for three different satellites were subjected to the Herrick-Gibbs technique to determine a series of midpoint velocity vectors throughout the observation time and, thence,

the resulting osculating elements. Time spacing between each of the three position vectors selected for the Herrick-Gibbs technique was varied from 6 to 60 seconds. This increase in the time interval considerably reduced the variation of the midpoint velocity. The velocity vector is computed from three weighted position vectors, the weights being

$$d_i = G_i + \frac{H_i}{\gamma_i^3}$$

The time spacing is such that G_i is orders of magnitude greater than H_i/γ_i^3 which is the component that considers the effect of the Keplerian motion. Since both G_i and H_i are functions of time, this magnitude difference can be reduced by increasing the time spacing. The longest time spacing indicated from the Millstone sample data is approximately four minutes and even with this length of time, the G_i term suppresses the Keplerian motion effect of H_i/γ_i^3 .

The osculating elements were then fitted with unweighted polynomials, the order of which was predetermined by the variate difference method and the statistics of the residuals of these osculating elements were derived. The complete computation procedure from radar parameters - osculating elements - selection of degree of polynomial and statistics of residuals has been programmed for the IBM 7090 in Fortran language.

The effect of increasing the time spacing of the data had the effect of decreasing the standard deviation of the osculating elements, shown in Figure 2-1 for three different satellite passes. The values of kurtosis varied between 2 and 3 and the semi-major axis indicated slight skewness to the right, while the inclination and eccentricity exhibited the opposite effect, see Figure 2-2, 2-3, and 2-4. Whereas, the cross correlation for the radar parameter could be considered negligible for the given sample size, some of the osculating elements have a strong correlation, particularly the semi-major axis, eccentricity and inclination angle.

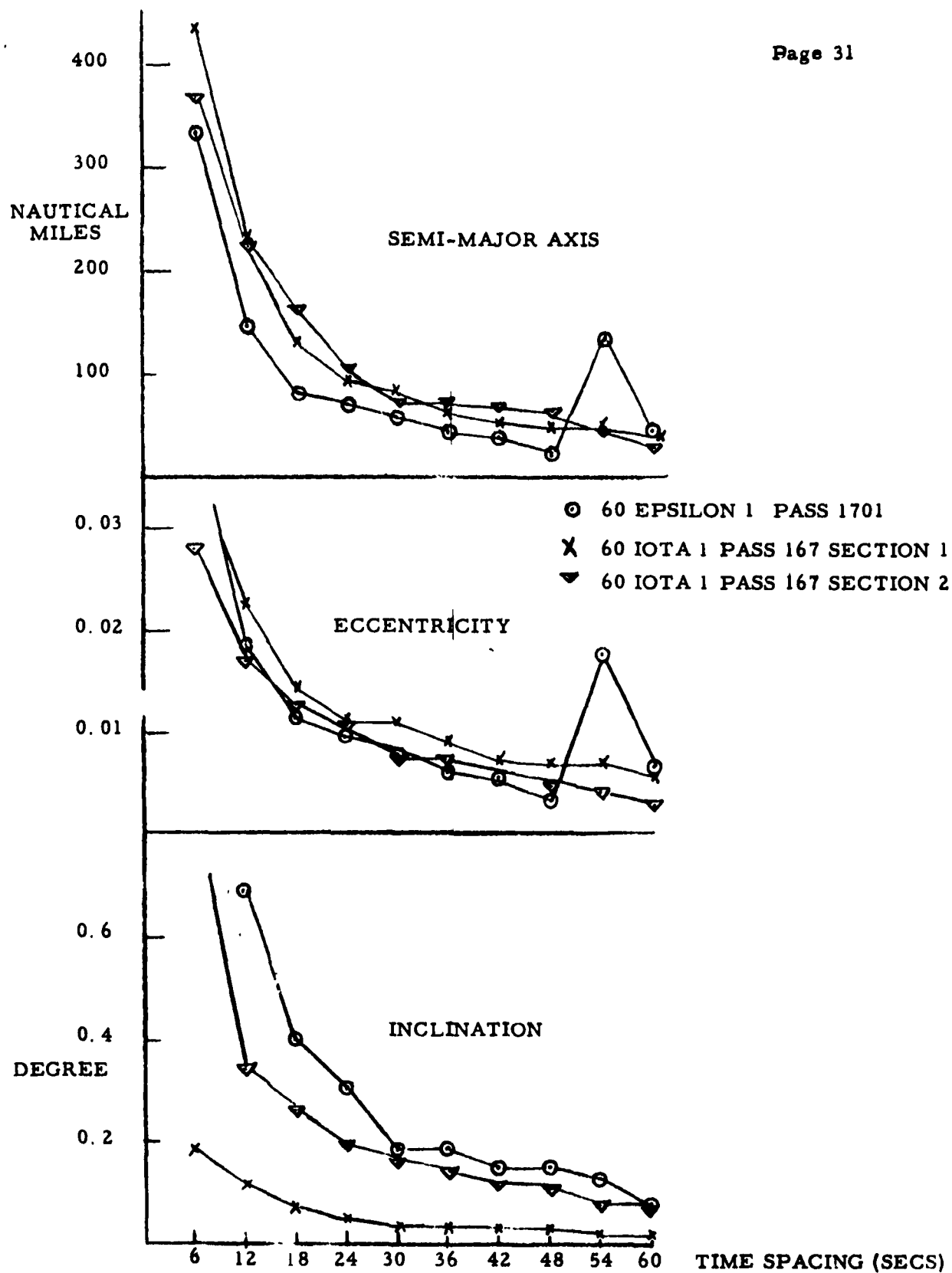


Figure 2-1. Standard deviation behavior of orbital elements for different time spacing.

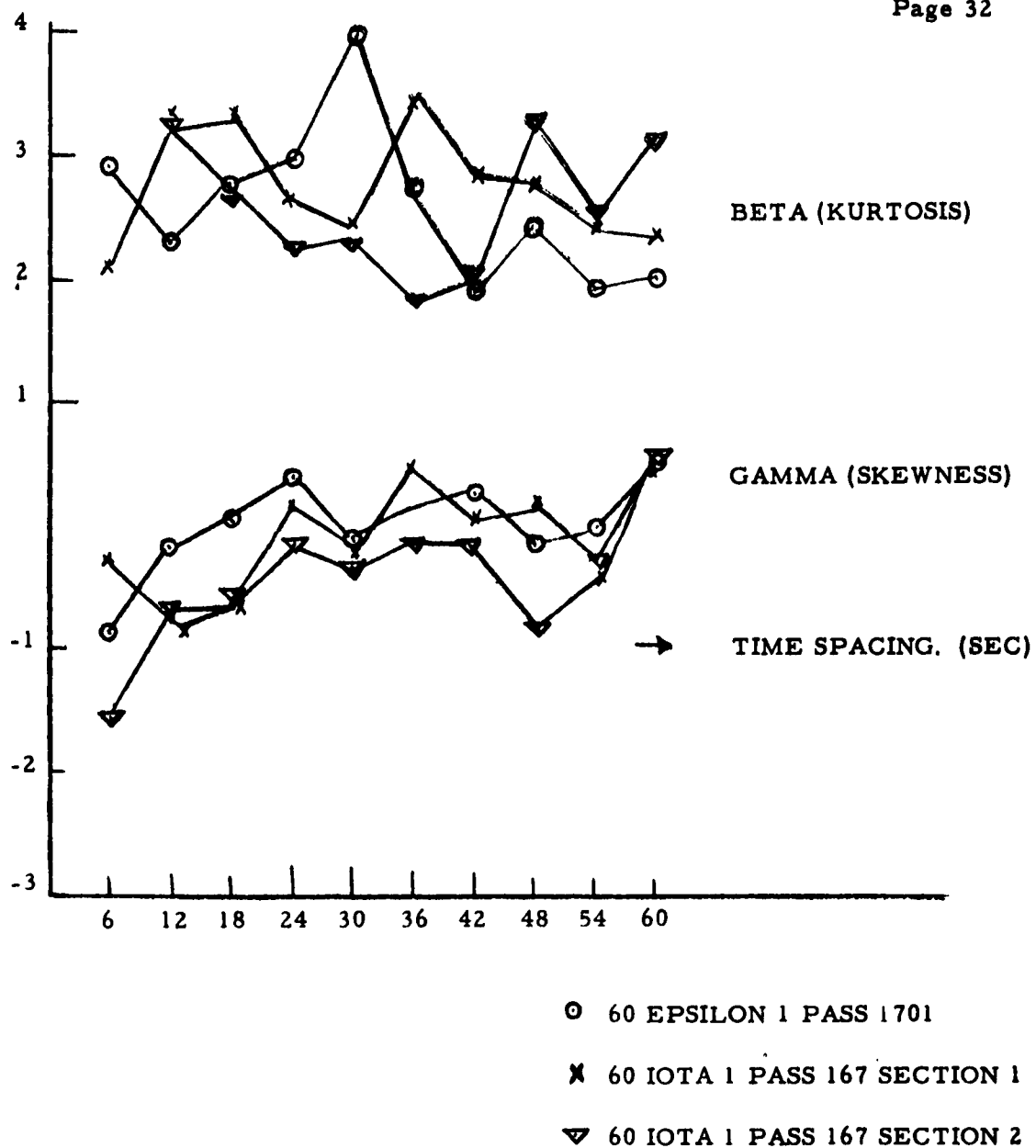


Figure 2-2. Kurtosis and skewness behavior of eccentricity for different time spacing.

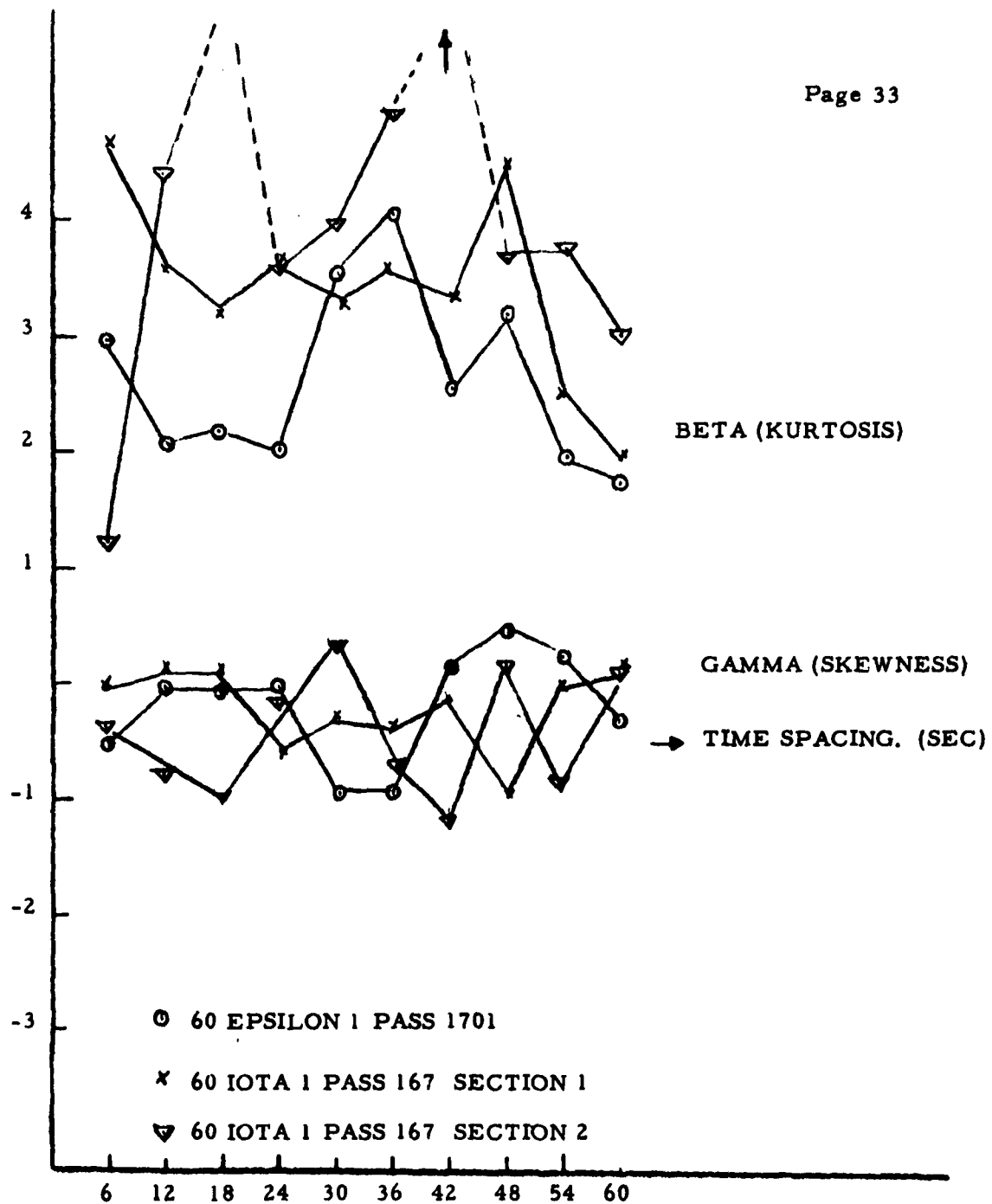


Figure 2-3. Kurtosis and skewness behavior of inclination for different time spacing.

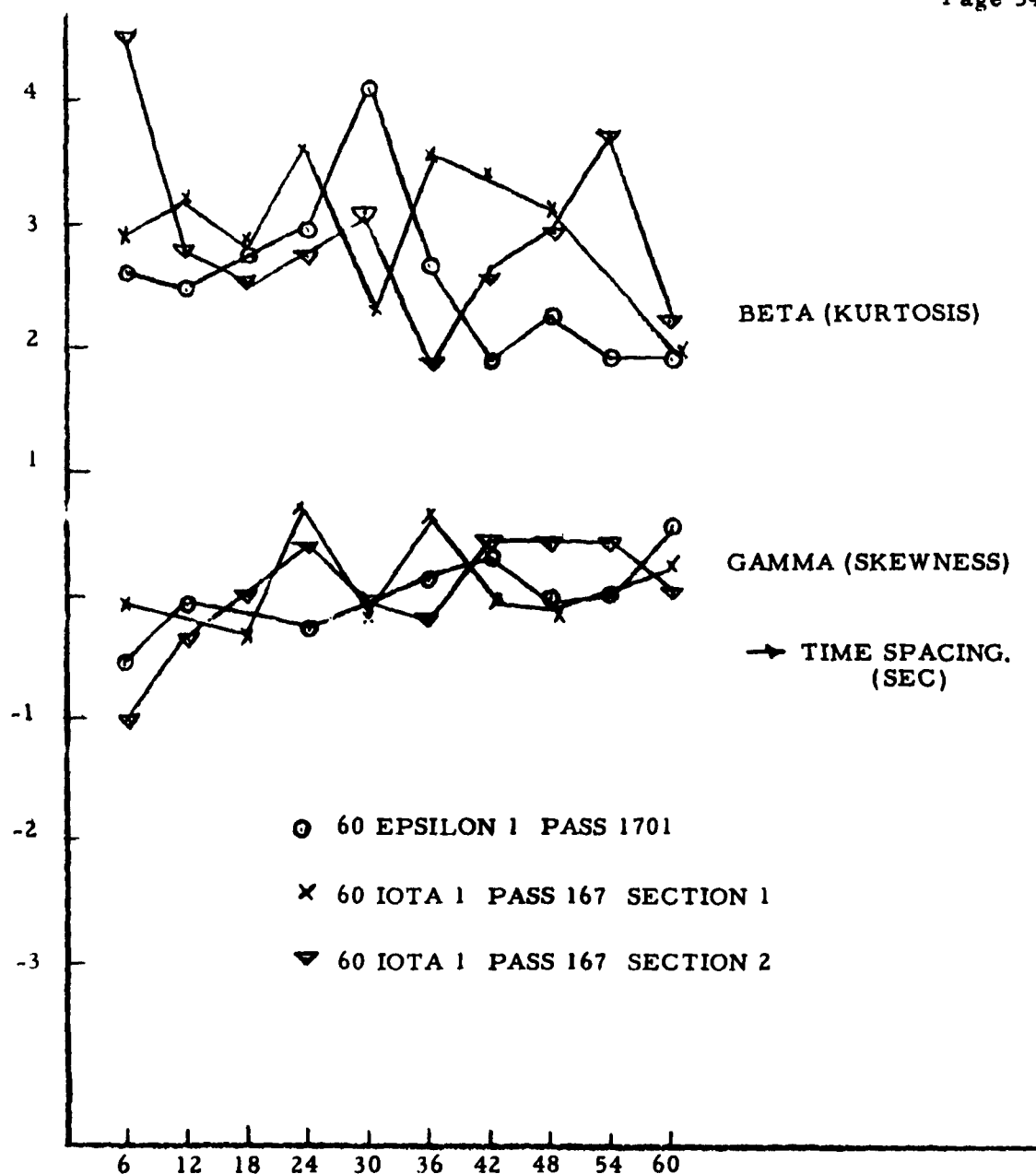


Figure 2-4. Kurtosis and skewness behavior of semi-major axis for different time spacing

The partial correlation coefficients for the various time spacings appeared to be relatively stable and show good agreement with the simple cross correlation for both the radar parameters and the orbital elements. The double sets of values shown for 1960 Epsilon 1 are due to the correlation between two parameters, with one of the other two parameters fixed.

1960 EPSILON 1 PASS 1701

Radar Parameters

Covariance Matrix

	<u>Elevation</u>	<u>Azimuth</u>	<u>Range</u>	<u>Range Rate</u>
Elevation	0.007837	-0.012349	-0.021095	0.000749
Azimuth	-0.012349	0.512653	0.041795	-0.066545
Range	-0.021095	0.041795	4.232353	-0.020629
Range Rate	0.000749	-0.066545	-0.020629	0.010595

Gross Correlation Coefficients

Elevation	1.000000	-0.194834	-0.111831	0.082151
Azimuth	-0.194834	1.000000	0.028374	-0.902930
Range	-0.115831	0.028374	1.000000	-0.097416
Range Rate	0.082151	-0.902930	-0.097416	1.000000

Partial Correlation Coefficients

Elevation	1	-0.192923	-0.112503	-0.22441
	1	-0.281689	-0.108711	0.071689
Azimuth	-0.192923	1	0.005960	-0.907320
	-0.281689	1	-0.139303	-0.904832
Range	-0.112503	0.005960	1	-0.088797
	-0.108711	-0.139303	1	-0.167118
Range Rate	-0.22441	-0.907320	-0.088797	1
	0.071689	-0.904832	-0.107118	1

1960 EPSILON 1 PASS 1701 (Continued)

Orbital Element (Time Spacing Bases)

Covariance Matrix

	Semi-major Axis (a)	Eccentricity (e)	Inclination Angle (i)	Ascension Angle Ω	Nodal Angle ω	Perigee Time τ_0
a	2154.8599	0.295694	1.433873	0.899188	14.1333	51.9656
e	0.2957	0.000041	0.000191	0.000116	0.0018	0.0079
i	1.4339	0.000191	0.005131	0.005536	0.0625	-0.0004
Ω	0.8992	0.000116	0.005536	0.006816	0.0543	-0.0481
ω	14.1333	0.001762	0.062502	0.054350	2.9182	-6.8399
τ_0	51.9656	0.007920	-0.000399	-0.048139	-6.1399	164.2565

Cross Correlation Coefficients

a	1.0000	0.999795	0.431223	0.234632	0.1782	0.0873
e	0.9998	1.000000	0.418186	0.221133	0.1619	0.0970
i	0.4312	0.418186	1.000000	0.936210	0.5108	-0.0004
Ω	0.2346	0.221133	0.936210	1.000000	0.3854	-0.0455
ω	0.1782	0.161902	0.510785	0.385379	1.0000	-0.3124
τ_0	0.0873	0.096997	-0.000435	-0.045497	-0.3124	1.0000

SECTION 3

CONCLUSIONS

This study has shown that the differential correction smoothing procedure is a solution of the generalized least square problem (Ref. 2). The residuals obtained by fitting the observation with polynomial plus an autoregressive scheme (Ref. 10) are uncorrelated and the standard deviations of the radar parameters with the autoregressive scheme are less than those obtained with polynomial fitting alone.

In the process of analyzing the behavior of the orbital elements it was noted that the coefficient of skewness exhibits a degree of consistency for different satellites and sensors. Furthermore it has been possible to derive the variance of any orbital elements, as functions of the variance of the observations, the time of observation and description of the motion of the satellite. It has been shown that the variance of any orbital element is a minimum if the sum of the variances of the observations at a given time is a minimum for a specific computation procedure of the orbital element from observations. This indicates that minimum variance orbital estimations are only relative minimums. The behavior of the variance and skewness suggests that the computational procedure introduced a bias. The latter, in particular, cannot be traced to a round-off error.

The problem of getting accurate position prediction of artificial satellites is one of interplay between the quality of observations and the analytical description of the motion of the satellite. How to trade these factors depends upon the time allowed for deriving the orbit from observations.

The analytical expression showing this interplay has been derived and represents the natural starting point for (1) an accurate prediction scheme and (2) updating or intermixing criteria.

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SYMBOLS USED

$\rho(t_i)$	Range at time t_i
$A(t_i)$	Azimuth at time t_i
$E(t_i)$	Elevation at time t_i
ϕ	Site latitude
λ	Site longitude
t_i	Time
(x, y, z)	A geocentric coordinate represented vector (x, y, z) in topocentric coordinates
$\underline{r}(t_i)$	A vector from earth's center to a point on orbit (at t_i)
R	Earth's radius
$\tau_1 \tau_2$	Time increments
$\dot{\underline{r}}(t_i)$	Rate of change of $\underline{r}(t_i)$
d_i, G_i, H_i	Defined in text, functions of
\underline{s}^2	Defined in text
P	Defined in text
$\underline{U}, \underline{V}, \underline{W}$	Defined in text (vectors)
a	Semi-latus rectum of orbit
e	Eccentricity of orbit
v	True anomaly of orbit
\underline{a}	A vector, defined in text
ω	The angle of perigee
$\underline{N}, \underline{M}$	Vectors defined in text
G	Gravitational constant

E	Eccentric anomaly of orbit
T	A constant of the orbit
Ω	The angle of the ascending node
I	The inclination of the orbit
$\underline{I}, \underline{J}, \underline{K}$	Unit vectors, defined in text
$E(x)$	The expectation of X
i_{ρ}^{μ}	The "ith" moment of P about its mean, $i=2, 3, 4$
μ_{ρ}	The mean of P
σ_{xy}	$E(x - \bar{x})(y - \bar{y})$
γ_{xyz}	$E(x - \bar{x})(y - \bar{y})(z - \bar{z}) / \sigma_x \sigma_y \sigma_z$
$\Gamma_{xyz\omega}$	$E(x - \bar{x})(y - \bar{y})(z - \bar{z})(\omega - \bar{\omega}) / \sigma_x \sigma_y \sigma_z \sigma_{\omega}$
$\underline{A}, \underline{B}$	Matrices, 5×3
a_{ij}	An element of $\underline{A} \underline{B}$
C_{xy}	$E(x - \bar{x})(y - \bar{y})$
C_{xyz}	$E(x - \bar{x})(y - \bar{y})(z - \bar{z})$
$C_{xyz\omega}$	$E(x - \bar{x})(y - \bar{y})(z - \bar{z})(\omega - \bar{\omega})$

APPENDIX A

POLAR VERSUS CARTESIAN SMOOTHING

Figure A-1 is a rough graph of the fourth degree curve of best fit to ten minutes of Millstone azimuth data. It is important to note that although the observations cluster around a smooth curve, this curve differs considerably from the fitting curve. This is because radar azimuth and elevation, and to a lesser extent, range, do not behave like low order polynomials for any length of time. To adequately represent the observations by a fourth degree polynomial, it is necessary to restrict the data to a much smaller interval, perhaps, one to two minutes. At the Millstone repetition rate, this allows only ten to twenty pulses to be smoothed. As it will be seen, a larger number of pulses is needed if the resulting observational errors are to have any statistical meaning.

Although the radar coordinates do not behave like low order polynomials, the cartesian inertial coordinates of the satellite do. The amount by which these coordinates deviate from a polynomial of degree p , during a time interval of length $2T$ is certainly less than the following expression:

$$(\text{Dev})_p < \frac{T^{p+1}}{(p+1)!} \left\| \underline{r}^{(p+1)} \right\|_{\max} \quad (\text{A-1})$$

where $\underline{r}^{(p+1)}$ is the $p+1^{\text{st}}$ derivative of the geocentric position vector, the double bars indicate its magnitude, and "max" refers to the maximum value of this magnitude over the time interval. Equation (A-1) is an application of Taylor's theorem.

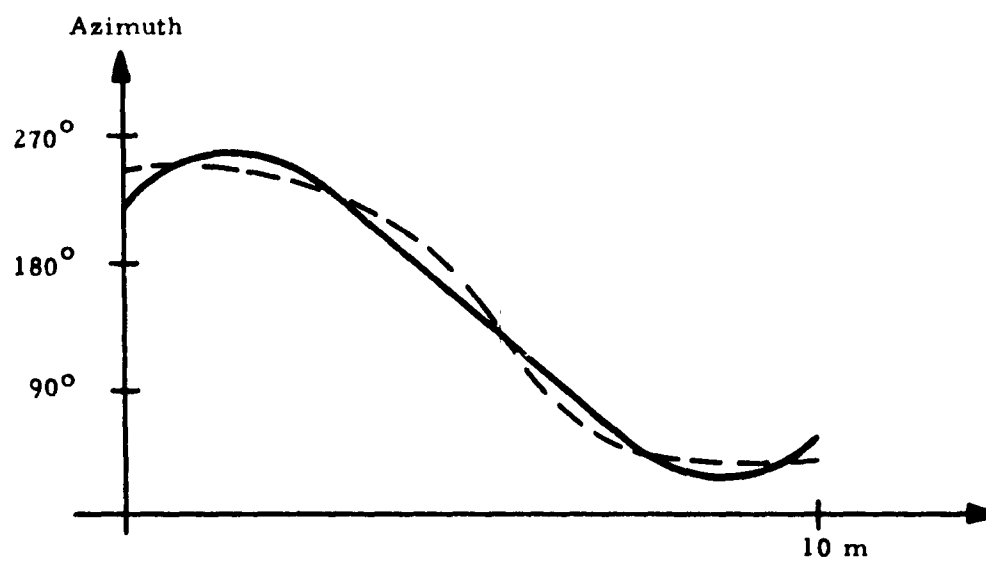


Figure A-1. Millstone azimuth observations (dashes) and best fitting fourth degree polynomial (curve)

In Table A-1, numerical bounds are found on the quantities $\left\| \underline{r}^{(p+1)} \right\|_{\max}$ by noting that (1) these quantities may be calculated without significant error assuming a Keplerian ellipse, and (2) on this ellipse the derivatives are maximized at perigee. The region of a-e space indicated in the third column includes most reasonable and stable satellite orbits. All formulas and numbers are in canonical units.

Table A-1

Magnitude of first few position derivatives is evaluated at perigee and numerical bounds on these magnitudes.

p	$\left\ \underline{r}^{(p)} \right\ $	$\left\ \underline{r}^{(p)} \right\ _{\max}$ over $a(1-e) > 1.05, e \leq 0.5$
0	$r_{\pi} = a(1-e)$	∞
1	$\sqrt{1+e} r_{\pi}^{-0.5}$	1.2
2	$r_{\pi}^{-2.0}$	0.9
3	$\sqrt{1+e} r_{\pi}^{-3.5}$	1.0
4	$(1+3e) r_{\pi}^{-5.0}$	2.0
5	$\sqrt{1+e} (1+9e) r_{\pi}^{-6.5}$	4.8
6	$(1+24e+27e^2) r_{\pi}^{-8.0}$	13.0

The second or third column of Table A-1 may be used with Equation (A-1) depending upon whether the orbital elements are approximately known. Combining the third column with Equation (A-1), Table A-2 may be obtained. Table A-2 expresses the maximum values of the intervals over which the inertial coordinates deviate from a p^{th} degree polynomial by the given amounts.

Table A-2

2T versus p and $(\text{Dev})_p$

P	$(\text{Dev})_p < 1.1 \text{ km}$	$(\text{Dev})_p < 0.1 \text{ km}$	$(\text{Dev})_p < 0.01 \text{ km}$
0	0.205 sec	0.0205 sec	0.00205 sec
1	29.9 sec	9.56 sec	2.99 sec
2	157.00 sec	72.90 sec	33.8 sec
3	334.00 sec	188.00 sec	106.00 sec
4	529.00 sec	334.00 sec	210.00 sec
5	726.00 sec	499.00 sec	337.00 sec

Table A-2 shows that, if a 0.1-km deviation in the fit can be accepted, a fourth degree polynomial may be applied to 5-1/2 minutes of data.

Thus, the following procedure has been developed for the orbit removal from Millstone radar.

- (1) From each pass, select the middle seven minutes of data
 $(\text{Dev}_4 (7 \text{ min}) < 0.35 \text{ km})$.
- (2) Transform the ranges, azimuths, and elevations selected in Step (1) to Cartesian inertial coordinates by the well-known transformation.

- (3) Solve the three sets of likelihood equations for the selected records of x , y , z .
- (4) Use the solutions to the likelihood equations to evaluate the fourth degree polynomials for x , y , z at all the t_k points selected in Step (1).
- (5) Transform these polynomial points back to range, azimuth and elevation and subtract them from the corresponding observations.

One simplification of this scheme is possible. Step 2 may be replaced by a transformation to the cartesian non-inertial coordinate system:

$$\begin{aligned}x &= R \sin E \\y &= R \cos E \sin A \\z &= R \cos E \cos A\end{aligned}\tag{A-2}$$

This procedure is simpler than the inertial transformation. The justification of this simplification is given by noting the relationship between the usual inertial system and the system defined by equations (A-2) as shown in Figure A-2. The inertial system has its origin at the earth's center with a z -axis along the earth's north pole, an x -axis pointed to the first point of Aries (γ), and a y -axis completing the right-hand system. The topocentric system has its origin at the position of the radar; its x -axis points directly away from the earth's center; its z -axis points in the direction of true north. Let the vector position of the satellite in the two systems be r , and ρ , respectively, then the transformation is obtained:

$$\rho = -\epsilon_1 + ABr\tag{A-3}$$

The matrices A and B and the vector ϵ_1 are given by:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (A-4)$$

In Equation (A-4), α is the declination of the observer, the β is his right ascension at any time. By differentiating Equation (A-2):

$$\begin{aligned} \rho^{(5)} = & A B r^{(5)} + 5 A \dot{B} r^{(4)} + 10 A \ddot{B} r^{(3)} + 10 A \dot{B} \ddot{r}^{(3)} \\ & + 5 A \dot{B}^{(4)} \dot{r} + A \dot{B}^{(5)} r \end{aligned} \quad (A-5)$$

Noting that $B^{(p)}$ is equal to $\dot{\beta}^p$ times a suborthogonal matrix (a rotation plus a projection) then:

$$\begin{aligned} \left| \rho^{(5)} \right| & \leq \left| r^{(5)} \right| + 5 \dot{\beta} \left| r^{(4)} \right| + 10 \dot{\beta}^2 \left| r^{(3)} \right| + \dots \\ & \leq 4.8 + 5(0.0587) (2.0) + 10(0.0587)^2 (1.0) + \dots \end{aligned} \quad (A-6)$$

$$\left| \rho^{(5)} \right| \leq 5.4$$

This is a somewhat conservative bound, probably 4.8 is more realistic. In any case, a fourth degree polynomial fits the data equally well, regardless of whether the data are expressed in inertial or topocentric coordinates.

APPENDIX B

DISTORTION OF THE ERRORS

Since the polynomial coefficients estimated from a record are random variables, the calculated values of the polynomials at each t_k are also random variables. If these values have standard deviations as large as the deviations of the observations, the actual observational errors will be masked. The deviations of the calculated values are calculated here under two assumptions:

- (1) $\sigma_{x(t_n) x(t_j)} = \sigma_x^2 \delta_{jk}$. That is, the data have no serial correlation, and the same coordinate observed at different times has the same variance.
- (2) $t_{k+1} = t_k + \tau$. That is, the observations are equally spaced in time.

Assumption (1) is not valid, for even if it were true of range, azimuth, and elevation, it would not be true that the cartesian coordinates would have constant variances. Assumption (2) is approximately satisfied for any radar - in particular for Millstone, $\tau = 6$ seconds.

Given assumption (1), Equation (B-1) for the covariance matrix of the coefficients is valid (with σ_R^2 replaced by σ_x^2 , σ_y^2 or σ_z^2). Given assumption (2), the sums in the matrix may be calculated in closed form, and the matrix analytically inverted. The results of this inversion are shown in Table B-1. The left-hand column is the degree of the polynomial. The matrix on the right hand has $\sigma_{A_{i-1}}$, A_{j-1} in its i^{th} row and j^{th} column. The quantity T , as defined previously, is one-half the time

interval, and the polynomial coefficients are evaluated at the time corresponding to the midpoint of the interval.

$$\sigma_{AA^T} \begin{pmatrix} \sigma_{A_0}^2 & \sigma_{A_0 A_1} & \dots & \sigma_{A_0 A_4} \\ \sigma_{A_0 A_1} & \sigma_{A_1}^2 & \dots & \sigma_{A_2 A_4} \\ \dots & \dots & \dots & \dots \\ \sigma_{A_0 A_4} & \sigma_{A_1 A_4} & \dots & \sigma_{A_4}^2 \end{pmatrix} = \sigma_R^2 \begin{pmatrix} N & \Sigma t_k & \dots & \Sigma t_k^4 \\ \Sigma t_k & \Sigma t_k^2 & \dots & \Sigma t_k^5 \\ \dots & \dots & \dots & \dots \\ \Sigma t_k^4 & \Sigma t_k^5 & \dots & \Sigma t_k^6 \end{pmatrix}$$

These covariance coefficients are obtained from the maximum likelihood equation of observation that are random variables.

Table B-1

Approximate covariance matrix of midpoint polynomial coefficients versus degree of polynomial, as a function of number of points, N ; half-time interval, T ; and variance of observations, σ^2 .

P	σ_{AA^T}
0	$\frac{1}{N} \sigma^2$
1	$\frac{1}{N} \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 3/T^2 \end{pmatrix}$
2	$\frac{3\sigma^2}{4N} \begin{pmatrix} 3 & 0 & -5/T^2 \\ 0 & 4/T^2 & 0 \\ -5/T^2 & 0 & 15/T^4 \end{pmatrix}$

Table B-1 (Continued)

3	$\frac{3\sigma^2}{4N}$	$\begin{pmatrix} 3 & 0 & -5/T^2 & 0 \\ 0 & 25/T^2 & 0 & -35/T^4 \\ -5/T^2 & 0 & 15/T^4 & 0 \\ 0 & -35/T^4 & 0 & 175/3T^6 \end{pmatrix}$
4	$\frac{5\sigma^2}{64N}$	$\begin{pmatrix} 45 & 0 & -210/T^2 & 0 & 189/T^4 \\ 0 & 240/T^2 & 0 & -336/T^4 & 0 \\ -210/T^2 & 0 & 1764/T^4 & 0 & -1890/T^6 \\ 0 & -336/T^4 & 0 & 560/T^6 & 0 \\ 189/T^4 & 0 & -1890/T^6 & 0 & 2205/T^8 \end{pmatrix}$

The data in Table B-1 are inaccurate for $N < \sim 20$. For $N \geq 20$, they are in excellent agreement with the true matrices.

From these data, the variances of the polynomial points may be determined. For example, for the point:

$$x_4(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 \quad (\text{B-2})$$

The variance is obtained by the formula:

$$\sigma_{x_4}^2(t) = (1 \ t \ t^2 \ t^3 \ t^4) \sigma_{AA^T} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{pmatrix} \quad (\text{B-3})$$

as:

$$\sigma_{x_4}^2(t) = \frac{\sigma_x^2}{N} \left[3.52 - 14.06 \left(\frac{t}{T}\right)^2 + 114.84 \left(\frac{t}{T}\right)^4 - 251.56 \left(\frac{t}{T}\right)^6 + 171.95 \left(\frac{t}{T}\right)^8 \right] \quad (B-4)$$

For polynomials of lower degree, expressions similar to Equation (B-4) are found. These expressions are listed in Table B-2. In Figure B-1 the square roots of these expressions and the ratio of the corresponding standard deviations are plotted for several degrees and several values of N.

Table B-2

Ratio of variance of polynomial point to variance of observation, $\sigma_{x_p}^2(t)/\sigma_x^2$ for various degrees of polynomial, p, as a function of N and of the ratio of time from midpoint to half interval, $K = t/T$.

p	$\sigma_{x_p}^2(t)/\sigma_x^2$
0	σ^2/N
1	$\sigma^2/N (1+3K^2)$
2	$\sigma^2/N (2.25-4K^2+11.25K^4)$
3	$\sigma^2/N (2.25+11.25K^2-41.25K^4+43.75K^6)$
4	$\sigma^2/N (3.516-14.064K^2+114.844K^4-251.562K^6+71.953K^8)$

Figure B-1 indicates that with N=70 and fourth degree smoothing, up to 80 percent of the computed points have a standard deviation less than one-quarter of the observation deviation; furthermore, at the extreme

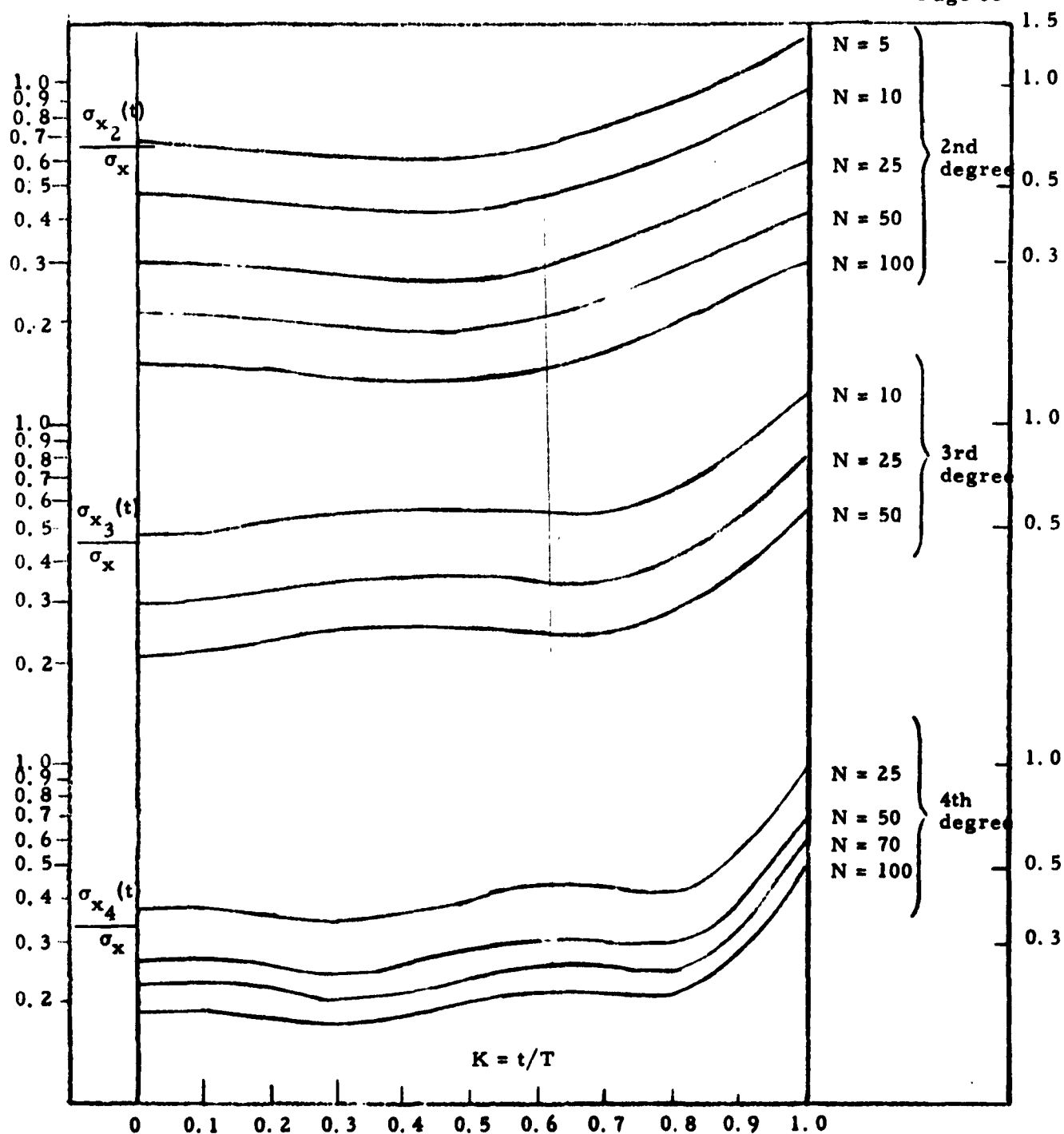


Figure B-1. Ratio of standard deviation of estimated polynomial point to standard deviation of observations, versus ratio of time from midpoint to half interval, for second, third, and fourth degree smoothing.

edges of the observation interval, this ratio is about three-fifths. However, since assumption (1) of the previous section - that the observation deviations in each coordinate are constant - is not true, the ratio may be somewhat higher than this value. For this reason, it is considered advisable to chop off the errors at about 80 percent of the way to the edges.

A smoothing procedure has thus been established to be applied to Millstone data covering seven or more minutes:

- (1) Select the middle seven minutes of data.
- (2) Transform range, azimuth and elevation to cartesian coordinates by Equations (A-2).
- (3) Find the coefficients of the fourth degree polynomials of closest fit to the three records obtained in Step (2) by solving the likelihood equations assuming observations are random variables.
- (4) Evaluate the three polynomials found in Step (3) at each of the 71 time points selected in Step (1).
- (5) Transform the polynomial points found in Step (4) to ranges, azimuths and elevations by inverting Equations (A-2) and subtract these values from the observed values of range, azimuth, and elevation.
- (6) Discard the first and last seven sets of differences leaving 57 "range errors," 57 "azimuth errors," and 57 "elevation errors."

This procedure has been devised for Millstone data, which are available at 6-second intervals, and which have standard deviations of about one to two kilometers. For different radars, a different procedure is necessary, but the formulas, tables, and graphs presented here should be sufficient to derive such a procedure.

APPENDIX C

CONFIDENCE BOUNDS FOR SAMPLE CORRELATION COEFFICIENTS AS A TEST FOR RANDOMNESS

In order to determine a test of randomness for a set of samples, an initial hypothesis $\rho = 0$ is taken. On the basis of the serial correlations of the sample, a procedure for testing randomness by employing the distribution function of the correlation coefficients is established. A 90-percent confidence curve on the correlation coefficient is shown in Figure C-1.

Given a sample of size m , it is desired to test the sample for randomness. One approach to this problem rests upon an examination of the serial correlations. The serial correlation coefficient is given by

$$r_k = \frac{\text{cov}(x_j, x_{j+k})}{\left[\text{var } x_j, \text{var } x_{j+k} \right]^{1/2}}, \quad k = 0, 1, 2 \dots \quad (\text{C-1})$$

If the elements x_j, x_{j+k} are separated into two groups, then the elements x_j and x_{j+k} can be considered as elements from separate samples, both samples being of the same size, n .

It is now assumed that the population has a bivariate normal distribution. That is, the samples x_j and x_{j+k} can be considered as coming from a bivariate normal population. The distribution function of r as given in Ref. 8 is:

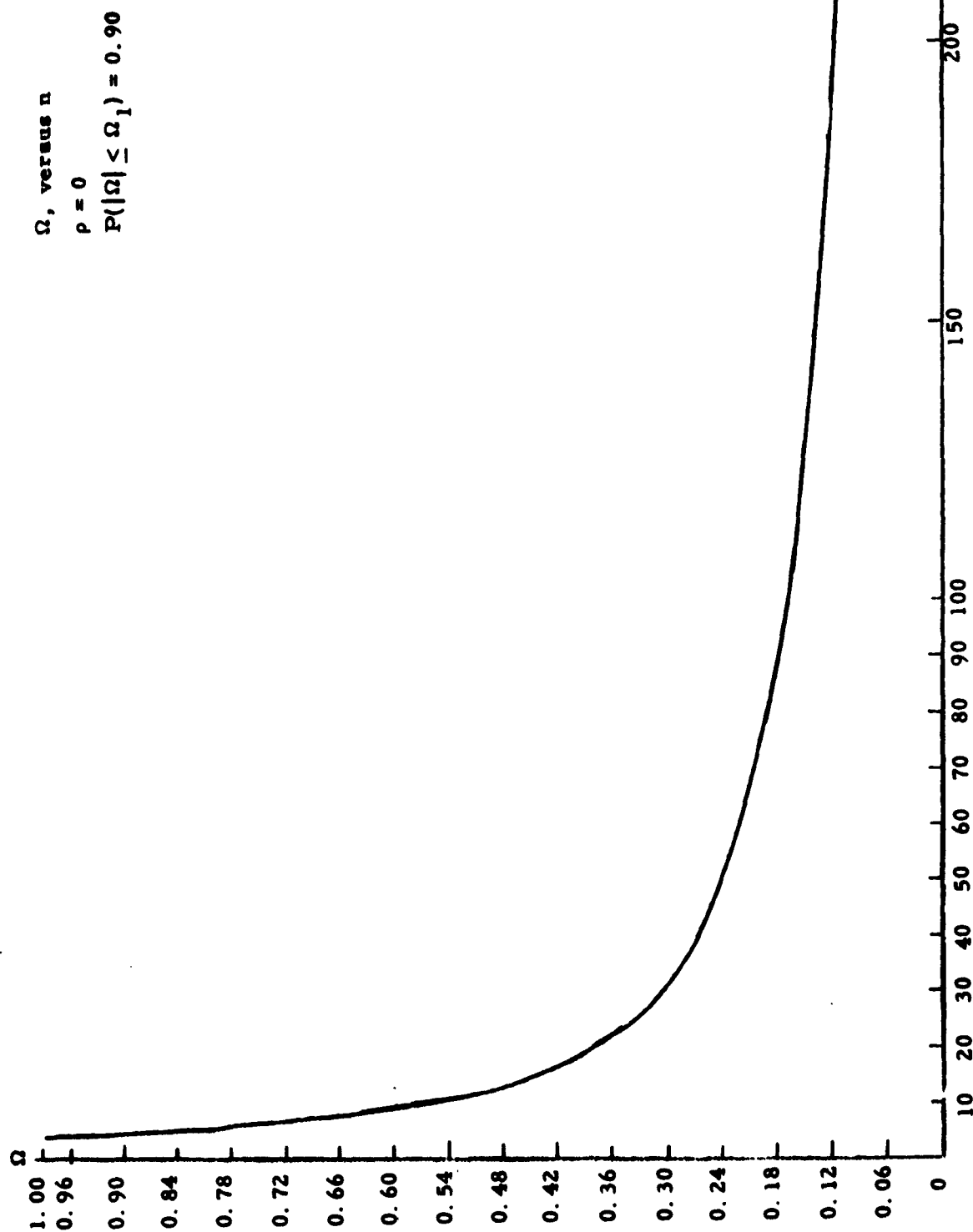


Figure C-1. 90 percent confidence curve for serial correlation coefficients for various sample - sizes.

$$q(r) = \frac{1 - \left[\frac{1 - \rho^2}{2} \right]^{n-1/2}}{\sqrt{\pi} \left[\frac{(n-1)/2}{2} \right] \cdot \left[\frac{n-2}{2} \right]} \left[1 - r^2 \right]^{\frac{n-4}{2}} \quad (C-2)$$

$$+ \sum_{k=0}^{\infty} \left[\frac{(2\rho r)^k}{k!} \right] \cdot \left[\frac{(n-1+k)/2}{2} \right], \quad -1 < r < 1, \quad \rho = \text{Er}$$

$$\text{for large } n, \quad \text{Er} = \rho, \quad \sigma_r^2 = \frac{(1 - \rho^2)^2}{n} \quad (C-3)$$

Tables of the distribution $q(r)$ can be found in Ref. 9.

A test for randomness under the initial hypothesis is now considered.

For $\rho = 0$, $q(r)$ reduces to

$$q(r) = \frac{\left[\frac{(n-1)/2}{2} \right]}{\sqrt{\pi} \left[\frac{(n-2)/2}{2} \right]} (1 - r^2)^{\frac{n-4}{2}} \quad (C-4)$$

Solving this expression for r ,

$$r = \pm \sqrt{1 - \frac{\left[\frac{\sqrt{\pi} q(r) \left[\frac{(n-2)/2}{2} \right]}{\left[\frac{(n-1)/2}{2} \right]} \right]^{\frac{2}{n-4}}}} \quad (C-5)$$

For a given $q(r)$, n , bounds can be established on r , $r_1 \leq r \leq r_2$, in order to test $H_0: \rho = 0$ with any level of confidence. The bounds may be rewritten as $|r| \leq r_1$ since r can be seen to be symmetric about zero.

To use the tables in Ref. 9, values of ρ and n are needed. For a given probability, a value of r can be determined. One such set of values is included in this report. These values are for a Two-Tailed Test of $H_0: \rho = 0$ with a probability of 0.90. The curve which follows the table is merely a graphical representation of the results with only the upper curve recorded since symmetry exists about zero.

For each serial correlation coefficient, the number of pairs of observations is plotted against r on the graph. If $|r|$ falls under the curve, it may be said that the sample is random with a probability of 0.90.

CRITICAL VALUES OF Ω FOR 90 PERCENT
CONFIDENCE BOUNDS

n	$\underline{\Omega}, P = 0.05$	$\overline{\Omega}, P = 0.95$
3	-0.975267	0.965267
4	-0.900	0.900
5	-0.805676	0.805676
6	-0.730231	0.730231
7	-0.670540	0.670540
8	-0.622746	0.622746
9	-0.583502	0.583502
10	-0.544231	0.544231
11	-0.522944	0.522944
12	-0.497620	0.497620
13	-0.477867	0.477867
14	-0.458413	0.458413
15	-0.441914	0.441914
16	-0.427845	0.427845
17	-0.413950	0.413950
18	-0.400026	0.400026
19	-0.390138	0.390138
20	-0.380474	0.380474
21	-0.370914	0.370914
22	-0.361411	0.361411
23	-0.351838	0.351838
24	-0.344748	0.344748
25	-0.338367	0.338367
50	-0.238043	0.238043
100	-0.170060	0.170060
200	-0.117231	0.117231
400	-0.083144	0.083144

APPENDIX D

DERIVATION OF THE ORBITAL ELEMENTS

This appendix summarizes the equations needed to obtain the orbital elements from observation data. The Herrick-Gibbs method is used in determining the position vector and velocity vector of an observed satellite.

D.1 DETERMINATION OF INERTIAL COORDINATES

In general, the observations of a satellite are made with reference to a given site of known latitude and longitude. To derive the orbit of this satellite the observed positions of the satellites are referred to as a geocentric inertial reference.

Let the earth's center be the origin of the geocentric inertial reference. The axis of the earth represents one of the axis, Z, of the reference. One of the axes, X, is in the equatorial plane and directed toward the first point in Aries or vernal equinox; the other, Y, is also in the equatorial plane and orthogonal to X. Let X', Y', Z' denote a topocentric reference system whose origin is the site. The X' axis coincides with the vertical through the site, Z' is the horizontal plane about the site and directed toward the north, and Y' forms an orthogonal system.

Let $\dot{\theta}$ be the angular rate of the earth's rotation in radians per second and let the earth be spherical of radius R. If X', Y', Z' is a point in the topocentric reference and X, Y, Z is a point in the inertial reference then:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos \lambda^* & -\sin \lambda^* & 0 \\ \sin \lambda^* & \cos \lambda^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

where ϕ is the site latitude, λ^* is the site longitude and $\lambda^* = \lambda + \dot{\theta} t$.

If the observations of a satellite about the site S are range, ρ , elevation, E , and azimuth, A ,

$$\begin{pmatrix} P x' (t) \\ P y' (t) \\ P z' (t) \end{pmatrix} = \begin{pmatrix} \rho \sin E \\ \rho \cos E \sin A \\ \rho \cos E \cos A \end{pmatrix}$$

then the position of the satellite in the inertial coordinate is represented by

$$\underline{r}(t_1) = \begin{pmatrix} X(t_1) \\ Y(t_1) \\ Z(t_1) \end{pmatrix} = \begin{pmatrix} \cos \lambda^* & -\sin \lambda^* & 0 \\ \sin \lambda^* & \cos \lambda^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{pmatrix} \begin{pmatrix} \rho(t_1) \sin E(t_1) + R \\ \rho(t_1) \cos E(t_1) \sin A(t_1) \\ \rho(t_1) \cos E(t_1) \cos A(t_1) \end{pmatrix}$$

where t_1 denotes the time of the observation

D.2 DETERMINATION OF POSITION AND VELOCITY VECTOR

From observations through vector positions, $\underline{r}(t_1)$, $\underline{r}(t_2)$, $\underline{r}(t_3)$ are determined using the transformations in subsection D.1. These positions are determined at approximately equal intervals of time. With these vectors $\underline{r}(t_2)$ and $\dot{\underline{r}}(t_2)$ are derived using the Herrick-Gibbs method.

$$\underline{r}(t_2) = \underline{r}(t_2)$$

$$\dot{\underline{r}}(t_2) = -d_1 \underline{r}(t_1) + d_2 \underline{r}(t_2) + d_3 \underline{r}(t_3)$$

$$\text{where } d_i = G_i + H_i [\underline{r}(t_i)]^{-3}$$

$$\text{and } H_1 = \tau_2/12, H_2 = (\tau_2 - \tau_1)/12, H_3 = \tau_1/12$$

$$G_1 = \tau_2^2 [\tau_1 \tau_2 (\tau_1 + \tau_2)]^{-1}, G_2 = (\tau_2^2 - \tau_1^2) [\tau_1 \tau_2 (\tau_1 + \tau_2)]^{-1},$$

$$G_3 = \tau_1^2 [\tau_1 \tau_2 (\tau_1 + \tau_2)]^{-1}$$

$$\text{where } \tau_1 = t_2 - t_1, \text{ and } \tau_2 = t_3 - t_2$$

D.3 DETERMINATION OF ORBITAL ELEMENTS

With the values of $\underline{r}(t_2)$ and $\dot{\underline{r}}(t_2)$ the orbital elements are determined as follows:

$$r^2 = \underline{r} \cdot \underline{r}$$

$$s^2 = \dot{\underline{r}} \cdot \dot{\underline{r}}$$

$$\dot{r} = \underline{r} \cdot \dot{\underline{r}} / r$$

$$\underline{U} = \underline{r} |\underline{r}|^{-1}$$

$$\underline{V} = \frac{\underline{r} \dot{\underline{r}} - \dot{\underline{r}} \underline{r}}{\sqrt{r^2 (s^2 - \dot{r}^2)}}$$

$$\underline{W} = \underline{U} \times \underline{V}$$

$$\frac{1}{a} = \frac{2}{|r|} - \frac{\dot{s}^2}{G}$$

$$e^2 = 1 - \frac{r^2 (\dot{s}^2 - \dot{r}^2)}{a G}$$

$$e \sin v = \frac{\dot{r}}{G} \sqrt{\dot{r}^2 (s^2 - r^2)}$$

$$e \cos v = \frac{\sqrt{(s^2 - r^2)}}{G} - 1$$

$$v = \tan^{-1} \left[\frac{e \sin v}{e \cos v} \right]$$

$$\underline{a} = \begin{bmatrix} U_x e \cos v - V_x e \sin v \\ U_y e \cos v - V_y e \sin v \\ U_z e \cos v - V_z e \sin v \end{bmatrix}$$

$$a_{xn} = (-a_x W_y + a_y W_x) / \sqrt{1 - W_z^2}$$

$$a_{yn} = (-a_x W_x W_z - a_y W_y W_z) / \sqrt{1 - W_z^2} + a_z \sqrt{1 - W_z^2}$$

$$\omega = \tan^{-1} \left[\frac{a_{yn}}{a_{xn}} \right], \quad u = \omega + v$$

$$\begin{aligned} & U_x \cos u - V_x \sin u \\ \underline{N} = & U_y \cos u - V_y \sin u \\ & U_z \cos u - V_z \sin u \end{aligned}$$

$$\begin{aligned} & - U_x \sin u + V_x \cos u \\ M = & U_y \sin u + V_x \cos u \\ & U_z \sin u + V_z \cos u \end{aligned}$$

$$\sin (V-E) = \frac{r G}{r^2 (\dot{s}^2 - \dot{r}^2)} \left[\frac{(e \sin v) (e \cos v)}{1 + \sqrt{1 - e^2}} + e \sin v \right]$$

$$\cos (v-E) = \frac{r}{r^2 (\dot{s}^2 - \dot{r}^2)} \left[\frac{1 - (e \sin v)^2}{1 + \sqrt{1 - e^2}} + e \cos v \right]$$

$$v-E = \tan^{-1} \left[\frac{\sin (v - E)}{\cos (v - E)} \right]$$

$$e \sin E = \frac{r \dot{r}}{\sqrt{G a}} \sqrt{\frac{G}{a^3}} \quad (t - T) = E - e \sin E$$

$$T = \sqrt{\frac{a^3}{G}} (E - e \sin E) + t_2$$

$$\Omega = \tan^{-1} \left[\frac{\underline{N} \times \underline{I}}{\underline{N} \cdot \underline{I}} \right] \quad \underline{I} = (1, 0, 0)$$

$$i = \tan^{-1} \left[\frac{\underline{W} \times \underline{K}}{\underline{W} \cdot \underline{K}} \right] \quad \underline{K} = (0, 0, 1)$$

Thus the six orbital elements are defined, namely

$$[a, e, i, \omega, \Omega, T]$$

APPENDIX E

THE MOMENTS OF PREDICTED POSITION ERROR FROM MOMENTS OF ORBITAL ELEMENT ERRORS

This appendix presents the method for obtaining the statistics of the error in the predicted position of a satellite as a function of the statistics of the error in orbital elements.

E.1. CALCULATION OF INERTIAL GEOCENTRIC COORDINATES FROM ORBITAL PARAMETER

E.1.1 STATEMENT OF THE PROBLEM

The determination of ephemeris of particular satellites from radar data which are subject to various errors will give predicted parameters subject to errors. The inputs are the orbital elements, which have assumed moments and comoments. The geocentric and topocentric inertial coordinates are then derived with their respective moment distributions. The initial orbital information was considered to be non-independent data, and the analysis utilized the first four moments to derive formulas for the corresponding moments of the geocentric inertial coordinates. These formulas indicate the procedure for the computation of such terms, but due to their length they were approximated to include only powers of order $< \sigma^2$ as they appear in Equations E.2.12, E.2.13, and E.2.14. The required partial differentials for substitution into these equations were then derived. The geocentric inertial coordinates are then transformed to topocentric coordinates and subsequently into radar parameters. The process for the computation of these error moments is an identical procedure as previously employed. The additional required partial differentials for the radar parameter computation have also been evaluated.

The calculation of the inertial geocentric coordinates from the orbital elements assuming a Keplerian system, and no orbital errors are outlined here. From the initial orbital elements $(a, e, \Omega_0, \omega, i, \tau_0)$ are geocentric inertial cartesian coordinates are derived. The "ox" axis is assumed to point to the first point in Aries through the Greenwich meridian at time τ_0 , the "oz" axis is through the north pole and "oy" completes the right-handed system.

Assuming a Keplerian system the magnitude of the radius vector at any given time τ is given by

$$r = \frac{a(1-e^2)}{1+e \cos v} \quad E(1.1)$$

where v is the true anomaly which is related to the eccentric anomaly and time by

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \quad E(1.2)$$

and

$$\sqrt{\frac{G}{a^3}} (\tau - \tau_0) = E - e \sin E \quad E(1.3)$$

From Equation E. 1. 1 the geocentric inertial coordinates can be obtained

$$\begin{aligned} x &= r \left\{ \cos \Omega_0 \cos (\omega_0 + v) - \sin \Omega_0 \sin (\omega_0 + v) \cos i \right\} \\ x &= r \left\{ \sin \Omega_0 \cos (\omega_0 + v) + \cos \Omega_0 \sin (\omega_0 + v) \cos i \right\} \\ z &= r \left\{ \sin (\omega_0 + v) \sin i \right\} \end{aligned} \quad E(1.4)$$

The ephemeris of the topocentric cartesian coordinates for a given radar site at latitude ϕ_s and longitude λ_s is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \begin{pmatrix} \cos \phi_s & 0 & \sin \phi_s \\ 0 & 1 & 0 \\ -\sin \phi_s & 0 & \cos \phi_s \end{pmatrix} \begin{pmatrix} \cos (2(t) + \lambda_s), \sin (\omega(t) + \lambda_s), 0 \\ -\sin (\omega(t) + \lambda_s), \cos (\omega(t) + \lambda_s), 0 \\ 0 & & 0 \ 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} R_o \\ 0 \\ 0 \end{pmatrix}$$

E (1.5)

From the topocentric coordinates the radar look angles and ranges are computed

$$\begin{aligned} \rho &= \sqrt{x'^2 + y'^2 + z'^2} \\ Az &= \tan^{-1} (y'/z') \\ El &= \cos^{-1} \frac{\sqrt{z'^2 + y'^2}}{e} \end{aligned} \quad E (1.6)$$

E.2 ERROR ANALYSIS

The geocentric inertial coordinates can be represented by a function of 6 orbital elements.

$$X_i = h^i (W_1, W_2, W_3, W_4, W_5, W_6) \quad E (2.1)$$

where:

$$\begin{aligned} X_1 &= X \\ X_2 &= Y \\ X_3 &= Z \end{aligned}$$

and

$$\begin{aligned} W_1 &= a \\ W_2 &= e \\ W_3 &= i \\ W_4 &= \Omega_o \\ W_5 &= \omega_o \\ W_6 &= \tau \end{aligned}$$

E (2.2)

Equation E.2.1 is represented by a Taylor's series about the mean value of the orbital elements.

$$\begin{aligned}
X_i = & h^i (\bar{w}_1 \dots \bar{w}_6) + \sum h^i_a w_a \\
& + \frac{1}{2} \sum h^i_{aa} w_a^2 + \sum^* h^i_{ab} w_a w_b \\
& + \frac{1}{6} \sum h^i_{aaa} w_a^3 + \frac{1}{2} \sum^* h^i_{aab} w_a^2 w_b + \sum^* h^i_{abc} w_a w_b w_c \\
& + \frac{1}{24} \sum h^i_{aaaa} w_a^4 + \frac{1}{6} \sum^* h^i_{aaab} w_a^3 w_b + \frac{1}{4} \sum^* h^i_{aabb} w_a^2 w_b^2 \\
& + \frac{1}{2} \sum^* h^i_{aabc} w_a^2 w_b w_c + \sum^* h^i_{abcd} w_a w_b w_c w_d \\
& + \dots
\end{aligned}
\tag{E 2.3}$$

where

$$h^i_{aab} = \frac{\partial^3}{\partial w_a^2 \partial w_b} h^i(w_1, w_2, \dots, w_6)_{(0,0,\dots,0)}
\tag{E 2.4}$$

and \sum^* implies the summation of all the different terms of the form written after it, which can be obtained without identifying subscripts (Ref. 11).

The original non-independent orbital elements are all subjected to errors which are assumed to have certain distributions of which their first four moments and their co-moments are known.

By definition $\mu_a^i = E[(w_a - \bar{w}_a)^i]$

$$\sigma_{ab} = E[(w_a - \bar{w}_a)(w_b - \bar{w}_b)]$$

$$\gamma_{abc} = E[(w_a - \bar{w}_a)(w_b - \bar{w}_b)(w_c - \bar{w}_c)]$$

E (2.5)

and

$$\Gamma_{abcd} = E[(w_a - \bar{w}_a)(w_b - \bar{w}_b)(w_c - \bar{w}_c)(w_d - \bar{w}_d)]$$

The first four moments of the inertial coordinates can now be formulated for the statistically non-independent case. The terms of order $\geq \sigma^5$ have been neglected.

$$\begin{aligned}
 \text{average } x_i &= h^i (W_1 \dots W_6) \\
 &+ \frac{1}{2} \sum h^i_{aa} \sigma_a^2 + \sum^* h^i_{ab} \rho_{ab} \sigma_a \sigma_b \\
 &+ \frac{1}{6} \sum h^i_{aaa} \gamma_{aaa} \sigma_a^3 + \frac{1}{2} \sum^* h^i_{aab} \gamma_{aab} \sigma_a^2 \sigma_b \\
 &+ \sum^* h^i_{abc} \gamma_{abc} \sigma_a \sigma_b \sigma_c \\
 &+ \frac{1}{24} \sum h^i_{aaaa} \Gamma_{aaaa} \sigma_a^4 + \frac{1}{6} \sum^* h^i_{aaab} \Gamma_{aaab} \sigma_a^3 \sigma_b \\
 &+ \frac{1}{4} \sum^* h^i_{aabb} \Gamma_{aabb} \sigma_a^2 \sigma_b^2 + \frac{1}{2} \sum^* h^i_{aabc} \Gamma_{aabc} \sigma_a^2 \sigma_b \sigma_c \\
 &+ \sum^* h^i_{abcd} \Gamma_{abcd} \sigma_a \sigma_b \sigma_c \sigma_d \\
 &+ \text{terms of order } \geq \sigma^5
 \end{aligned}
 \tag{2.6a}$$

$$\begin{aligned}
 \text{variance } x_i &= \sum h^i_a h^i_a \sigma_a^2 + 2 \sum^* h^i_a h^i_b \rho_{ab} \sigma_a \sigma_b \\
 &+ \sum h^i_a h^i_{aa} \sigma_{aaa} \sigma_a^3 + \sum h^i_a h^i_{bb} \gamma_{abb} \sigma_a \sigma_b^2 \\
 &+ 2 \sum^* h^i_a h^i_{ab} \gamma_{aab} \sigma_a^2 \sigma_b + 2 \sum^* h^i_a h^i_{bc} \gamma_{abc} \sigma_a \sigma_b \sigma_c \\
 &+ \frac{1}{4} \sum h^{i2}_{aa} (\Gamma_{aaaa} - 1) \sigma_a^4 + \frac{1}{2} \sum^* h^{i2}_{aa} (\Gamma_{aabb} - 1) \sigma_a^2 \sigma_b^2 \\
 &+ \sum^* h^i_{aa} h^i_{ab} (\Gamma_{aab} - \rho_{ab}) \sigma_a^3 \sigma_b + \sum^* h^i_{aa} h^i_{bc} (\Gamma_{aabc} - \rho_{bc}) \sigma_a^2 \sigma_b \sigma_c \\
 &+ \sum^* h^{i2}_{ab} (\Gamma_{aabb} - \rho_{ab}^2) \sigma_a^2 \sigma_b^2 + 2 \sum^* h^i_{ab} h^i_{ac} (\Gamma_{aabc} - \rho_{ab} \rho_{ac}) \sigma_a \sigma_b \sigma_c
 \end{aligned}$$

$$\begin{aligned}
& +2 \sum^* h_{ab}^i h_{cd}^i (\Gamma_{abcd} - \rho_{ab} \rho_{cd}) \sigma_a \sigma_b \sigma_c \sigma_d \\
& + \frac{1}{3} \sum h_a^i h_{aaa}^i \Gamma_{aaaa} \sigma_a^4 + \frac{1}{3} \sum^* h_a^i h_{bbb}^i \Gamma_{abbb} \sigma_a \sigma_b^3 \\
& + \sum^* h_a^i h_{aab}^i \Gamma_{aaab} \sigma_a^3 \sigma_b + \sum^* h_a^i h_{abb}^i \Gamma_{aabb} \sigma_a^2 \sigma_b^2 \\
& + \sum^* h_a^i h_{bbc}^i \Gamma_{abbc} \sigma_a \sigma_b^2 \sigma_c + 2 \sum^* h_a^i h_{abc}^i \Gamma_{aabc} \sigma_a^2 \sigma_b \sigma_c \\
& + 2 \sum h_a^i h_{bcd}^i \Gamma_{abcd} \sigma_a \sigma_b \sigma_c \sigma_d \\
& + \text{terms of order } > \sigma^5
\end{aligned}$$

E (2.6b)

$$\begin{aligned}
\text{ske } x_i &= \sum h_a^{i3} \gamma_{aaa} \sigma_a^3 + 3 \sum^* h_a^{i2} h_b^i \gamma_{aab} \sigma_a^2 \sigma_b \\
& + 6 \sum^* h_a^i h_b^i h_c^i \gamma_{abc} \sigma_a \sigma_b \sigma_c + \frac{3}{2} \sum h_a^{i2} h_{aa}^i (\Gamma_{aaaa} - 1) \sigma_a^4 \\
& + \frac{3}{2} \sum h_a^{i2} h_{bb}^i (\Gamma_{aabb} - 1) \sigma_a^2 \sigma_b^2 + 3 \sum^* h_a^i h_b^i h_{aa}^i (\Gamma_{aaab} - \rho_{ab}) \sigma_a^3 \sigma_b \\
& + 3 \sum^* h_a^{i2} h_{bc}^i (\Gamma_{aabc} - \rho_{bc}) \sigma_a^2 \sigma_b \sigma_c + \sigma \sum^* h_a^i h_b^i h_{ab}^i (\Gamma_{aabb} - \rho_{ab}^2) \sigma_a^2 \sigma_b^2 \\
& + 6 \sum^* h_a^i h_b^i h_{ac}^i (\Gamma_{aabc} - \rho_{ab} \rho_{ac}) \sigma_a^2 \sigma_b \sigma_c \\
& + 6 \sum^* h_a^i h_b^i h_{cd}^i (\Gamma_{abcd} - \rho_{ab} \rho_{cd}) \sigma_a \sigma_b \sigma_c \sigma_d \\
& + \text{terms of order } > \sigma^5
\end{aligned}$$

E (2.6c)

$$\begin{aligned}
\text{elo } x_i &= \sum h_a^{i4} (\Gamma_{aaaa} - 3) \sigma_a^4 \\
& + 4 \sum^* h_a^{i3} h_b^i (\Gamma_{aaab} - 3 \rho_{ab}) \sigma_a^3 \sigma_b + 6 \sum^* h_a^{i2} h_b^{i2} (\Gamma_{aabb} - 2 \rho_{ab}^2 - 1) \sigma_a^2 \sigma_b^2 \\
& + 12 \sum^* h_a^{i2} h_b^i h_c^i (\Gamma_{aabc} - \rho_{bc} - 2 \rho_{ab} \rho_{ac}) \sigma_a^2 \sigma_b \sigma_c \\
& + 24 \sum^* h_a^i h_b^i h_c^i h_d^i (\Gamma_{abcd} - \rho_{ab} \rho_{cd} - \rho_{ac} \rho_{bd} - \rho_{ad} \rho_{bc}) \sigma_a \sigma_b \sigma_c \sigma_d \\
& + \text{terms of order } > \sigma^5
\end{aligned}$$

E (2.6d)

The co-moments for the non-independent case will result also in functions of the original distribution moments of the orbital elements. This has not been pursued to the same extent as the variance, etc., but only the co-variance has been considered for powers of second order. A sample derivation is indicated. Co-variance for non independent variables is

$$\sigma_{xy} = E ((x-\bar{x}) (y-\bar{y})) \quad E (2.7)$$

where x, y are represented by a truncated Taylor series.

$$\begin{aligned} x &= h (\bar{W}_1 \dots \bar{W}_6) + \sum h_a (W_a - \bar{W}_a) + \frac{1}{2} \sum h_{aa} (W_a - \bar{W}_a)^2 \\ &\quad + \sum^* h_{ab} (W_a - \bar{W}_a) (W_b - \bar{W}_b) \\ \text{and} \\ y &= g (\bar{W}_1 \dots \bar{W}_6) + \sum g_a (W_a - \bar{W}_a) + \frac{1}{2} \sum g_{aa} (W_a - \bar{W}_a)^2 \\ &\quad + \sum^* g_{ab} (W_a - \bar{W}_a) (W_b - \bar{W}_b) \end{aligned} \quad E(2.8)$$

From the equation E. 2.8 the average values for x and y are

$$\begin{aligned} x &= h (\bar{W}_1 \dots \bar{W}_6) + \frac{1}{2} \sum h_{aa} \sigma_a^2 + \sum^* h_{ab} \sigma_{ab} \\ y &= g (\bar{W}_1 \dots \bar{W}_6) + \frac{1}{2} \sum g_{aa} \sigma_a^2 + \sum^* g_{ab} \sigma_{ab} \end{aligned} \quad E(2.9)$$

therefore

$$\begin{aligned} xy &= h(\bar{W}_1 \dots \bar{W}_6) g (\bar{W}_1 \dots \bar{W}_6) + \frac{1}{2} h(\bar{W}_1 \dots \bar{W}_6) \sum g_{aa} \sigma_a^2 \\ &\quad + h (\bar{W}_1 \dots \bar{W}_6) \sum^* g_{ab} \sigma_{ab} + \frac{1}{2} g (\bar{W}_1 \dots \bar{W}_6) \sum h_{aa} \sigma_a^2 \\ &\quad + g (\bar{W}_1 \dots \bar{W}_6) \sum^* h_{ab} \sigma_{ab} \\ &\quad + \text{terms of order } > \sigma^2 \end{aligned} \quad E(2.10)$$

$$E(xy) = \left[\begin{aligned} &h(\bar{W}_1 \dots \bar{W}_6) g(\bar{W}_1 \dots \bar{W}_6) + h(\bar{W}_1 \dots \bar{W}_6) \sum g_a (W_a - \bar{W}_a) \\ &+ \frac{1}{2} g(\bar{W}_1 \dots \bar{W}_6) \sum g_{aa} (W_a - \bar{W}_a)^2 + h(\bar{W}_1 \dots \bar{W}_6) \sum^* g_{ab} (W_a - \bar{W}_a) (W_b - \bar{W}_b) \\ &+ g(\bar{W}_1 + \bar{W}_6) \sum h_a (W_a - \bar{W}_a) + \sum h_a (W_a - \bar{W}_a) \sum g_a (W_a - \bar{W}_a) \\ &+ \frac{1}{2} g(\bar{W}_1 \dots \bar{W}_6) \sum h_{aa} (W_a - \bar{W}_a)^2 \\ &+ g(\bar{W}_1 \dots \bar{W}_6) \sum^* h_{ab} (\bar{W}_a - \bar{W}_a) (W_b - \bar{W}_b) \end{aligned} \right]$$

$$\begin{aligned} E(xy) = & h(\bar{W}_1 \dots \bar{W}_6) g(\bar{W}_1 \dots \bar{W}_6) + \frac{1}{2} h(\bar{W}_1 \dots \bar{W}_6) \sum g_{aa} \sigma_{aa}^2 \\ & + h(\bar{W}_1 - \bar{W}_6) \sum^* g_{ab} \sigma_a \sigma_b + \sum_a h_a \sigma_a^2 + 2 \sum^* h_a g_b \\ & + \frac{1}{2} g(\bar{W}_1 \dots \bar{W}_6) \sum h_{aa} \sigma_{aa}^2 + g(\bar{W}_1 \dots \bar{W}_6) \sum^* h_{ab} \sigma_a \sigma_b \end{aligned}$$

E(2. 11)

Substitute equation E. 2. 11 and E. 2. 10 into equation E. 2. 7 and thus the co-variance becomes

$$\begin{aligned} \sigma_{xy} = & \sum g_a h_a \sigma_a^2 + 2 \sum^* h_{agb} \sigma_{ab} \\ & + \text{terms of order } > \sigma^2 \end{aligned}$$

E(2. 12)

Considering only the powers of order $< \sigma^2$ the variance can be written

$$\sigma_{xx} = \sum h_a^2 \sigma_a^2 + 2 \sum^* h_a h_b \sigma_{ab}$$

E(2. 13)

and the average $x_i = h^i(\bar{W}_1 \dots \bar{W}_6) + \frac{1}{2} \sum h_{aa}^i \sigma_a^2 + \sum^* h_{ab}^i \sigma_a \sigma_b$

E(2. 14)

From equation E. 2. 12 and E. 2. 13 the variance and co-variance of the geocentric inertial coordinates can be formulated.

The moments for the topocentric coordinate at a particular radar site are not computed from the known moments of the geocentric inertial coordinates. The topocentric coordinates are expressed as a Taylor's series about the mean value of the inertial coordinates.

$$x_i = h^i (\bar{X}\bar{Y}\bar{Z}) + \sum h_a^i W_a + \frac{1}{2} \sum h_{aa}^i W_a^2 + \sum h_{ab}^i W_a W_b \quad \text{E (2. 15)}$$

The average, variance and co-variance of the topocentric co-ordinates will be similar to those expressed in equations E. 2. 14, E. 2. 12, and E. 2. 13. The final transformation is from these topocentric co-ordinates to radar look angles and slant range as indicated by Equation E. 1. 6.

E. 3 EVALUATION OF PARTIAL DERIVATIVES FOR MOMENT FORMULATION

The results of the partial derivative required for the formulation of equation E. 2. 12, E. 2. 13, and E. 2. 14 will now be evaluated. As can be seen from the lengthy nature of each of the partial differentiation terms, the moments were not evaluated beyond terms requiring powers of order greater than 2.

$$\frac{\partial x}{\partial a} = \left(\frac{(1-e^2)}{1+e \cos v} + \frac{3e \sin v}{2(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} (\tau - \tau_0) \right) A + \frac{3}{2} \sqrt{\frac{G}{a^3}} \frac{(\tau - \tau_0) (1 + e \cos v)}{(1 - e^2)^{1/2}} \cdot \frac{dA}{dv}$$

$$\frac{\partial x}{\partial e} = a (3 \cos^2 v (e \cos v + 2) - \cos v) A + \frac{a \sin v (e \cos v + 2)}{(1 + e \cos v)} \frac{dA}{dv}$$

$$\frac{\partial x}{\partial \Omega} = \frac{a (1 - e^2)}{1 + e \cos v} \cdot \frac{dA}{d\Omega}$$

$$\frac{\partial x}{\partial \omega} = \frac{a (1 - e^2)}{1 + e \cos v} \cdot \frac{dA}{d\omega}$$

$$\frac{\partial x}{\partial i} = \frac{a (1 - e^2)}{1 + e \cos v} \cdot \frac{dA}{di}$$

$$\frac{\partial x}{\partial \tau} = \frac{a e \sin v}{(1 - e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot A - \frac{a (1 + e \cos v)}{(1 - e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot \frac{dA}{dv}$$

$$\text{where } A = \cos e \cos (\omega_0 + v) - \sin \Omega \sin (\omega_0 + v) \cos i$$

$$\begin{aligned} \frac{\partial y}{\partial a} &= \frac{(1 - e^2)}{(1 + e \cos v)} - \frac{3 e \sin v}{2 (1 - e^2)^{1/2}} \sqrt{\frac{G}{a^3}} (\tau - \tau_0) \cdot B \\ &\quad - \frac{3 (1 + e \cos v)}{2 (1 - e^2)^{1/2}} \sqrt{\frac{G}{a^3}} (\tau - \tau_0) \cdot \frac{dB}{dv} \end{aligned}$$

$$\begin{aligned} \frac{\partial y}{\partial e} &= \left[-\frac{a \cos v (1 + 2 e^2)}{(1 + e \cos v)^2} + a e \frac{(\sin^2 v - e^2 \sin^2 v - 1) (e \cos v + 2)}{(1 + e \cos v)^2} \right] \cdot B \\ &\quad + \frac{a (1 - e^2) \sin v (e \cos v + 2)}{(1 + e \cos v)} \cdot \frac{dB}{dv} \end{aligned}$$

$$\frac{\partial y}{\partial \Omega} = \frac{a (1 - e^2)}{(1 + e \cos v)} \cdot \frac{dB}{d\Omega}$$

$$\frac{\partial y}{\partial \omega} = \frac{a (1 - e^2)}{(1 + e \cos v)} \cdot \frac{dB}{d\omega}$$

$$\frac{\partial y}{\partial i} = \frac{a(1-e^2)}{(1+e \cos v)} \cdot \frac{dB}{di}$$

$$\frac{\partial y}{\partial \tau} = \frac{-ae \sin v}{(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot B - \frac{a(1+e \cos v)}{(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot \frac{dB}{dv}$$

$$\text{where } B = \sin \Omega \cos (\omega_0 + v) - \cos \Omega \sin (\omega_0 + v) \cos i$$

$$\begin{aligned} \frac{\partial z}{\partial a} = & \left\{ \frac{(1-e^2)}{(1+e \cos v)} - \frac{3e \sin v}{2(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} (\tau - \tau_0) \right\} \cdot C \\ & - \frac{3(1+e \cos v)}{2(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} (\tau - \tau_0) \cdot \frac{dC}{dv} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial e} = & \left\{ \frac{(-2ae)}{(1+e \cos v)} - \frac{a(1-e^2) \cos v}{(1+e \cos v)^2} + \frac{ea \sin^2 v (1-e^2)(e \cos v + 2)}{(1+e \cos v)} \right\} \\ & - \frac{a(1-e^2) \sin v (e \cos v + 2)}{(1+e \cos v)} \cdot \frac{dC}{dv} \end{aligned}$$

$$\frac{\partial z}{\partial \Omega} = 0$$

$$\frac{\partial z}{\partial \omega} = \frac{a(1-e^2)}{(1+e \cos v)} \cdot \frac{dc}{d\omega}$$

$$\frac{\partial z}{\partial i} = \frac{a(1-e^2)}{(1+e \cos v)} \cdot \frac{dc}{di}$$

$$\begin{aligned} \frac{\partial z}{\partial \tau} = & \frac{-ea \sin v}{(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot C \\ & - \frac{a(1+e \cos v)}{(1-e^2)^{1/2}} \sqrt{\frac{G}{a^3}} \cdot \frac{dc}{dv} \end{aligned}$$

$$\text{where } C = \sin (\omega_0 + v) \sin i$$

The ensuing partial derivatives are required for evaluation of the radar parameters from the moment distribution of the topocentric co-ordinates.

$$\frac{\partial \rho}{\partial x} = \frac{x'}{\rho}$$

$$\frac{\partial \rho}{\partial y} = \frac{y'}{\rho}$$

$$\frac{\partial \rho}{\partial z} = \frac{z'}{\rho}$$

$$\frac{\partial A_z}{\partial y'} = \frac{z'}{(z')^2 + (y')^2}$$

$$\frac{\partial E1}{\partial x} = - \sqrt{(y')^2 + (z')^2}$$

$$\frac{\partial E1}{\partial y} = - \frac{-x' y'}{\rho^2 \sqrt{(z')^2 + (y')^2}}$$

$$\frac{\partial E1}{\partial z} = \frac{-x' z'}{\rho^2 \sqrt{(z')^2 + (y')^2}}$$

These derivatives have to be substituted in equations (E. 2. 6) (E. 2. 12), (E. 2. 13), (E. 2. 14) to get the statistics of the moments of the prediction error.